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Escape over a potential barrier driven by colored noise: Large but finite correlation times

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The recent theory of Tsironis and Grigolini for the mean first-passage time from one metastable state to another of a bistable potential for long correlation times of the noise is extended to large but finite correlation times.

Recently, Tsironis and Grigolini (TG)[1,2] presented a new theory for the mean first-passage time from one metastable state to another of a bistable process driven by colored noise. Their theory is expected to be valid when the correlation time τ of the noise is extremely long ($\tau\to\infty$). TG specifically considered the system

$$\dot{x}=\alpha x-\beta x^3+\xi(t)\,, \tag{1}$$

wherein the fluctuations $\xi(t)$ are Gaussian and exponentially correlated:

$$\langle\xi(t)\xi(t')\rangle=\frac{D}{\tau}e^{-|t-t'|/\tau}. \tag{2}$$

Let us briefly restate the TG argument in a way convenient for our purposes. One can associate with the process (1) an instantaneous "potential"

$$V(x)=-\alpha\frac{x^2}{2}+\beta\frac{x^4}{4}-\xi x\,. \tag{3}$$

The number of extrema of this potential depends on the value of ξ. If $|\xi|<\xi_c\equiv(4\alpha^3/27\beta)^{1/2}$, then $V(x)$ has three extrema, a maximum corresponding to an unstable state and two minima corresponding to stable states. When $|\xi|=\xi_c$, two of these extrema merge into a single marginally stable state. If $|\xi|>\xi_c$, then there is only a single extremum (a minimum). These three cases are illustrated in Fig. 1.

For large τ ($\tau\to\infty$), TG argue that passage from one metastable state to the other occurs when the metastable state in which the process finds itself disappears. Thus, if the process is in the left-hand well ($x<0$), then the first passage to the right-hand well ($x>0$) occurs when $\xi(t)$ first reaches the value ξ_c. To calculate this passage time, one notes that the fluctuations $\xi(t)$ constitute an Ornstein-Uhlenbeck process whose evolution can be described by the Langevin equation

$$\dot{\xi}=-\frac{1}{\tau}\xi+\frac{1}{\tau}f(t)\,, \tag{4}$$

where $f(t)$ is Gaussian delta-correlated noise

$$\langle f(t)f(t')\rangle=2D\delta(t-t')\,. \tag{5}$$

The mean time for $\xi(t)$ to first reach ξ_c is obtained using standard methods[3] and for $\xi_c\tau/D\gg1$ is found to be

$$T=\frac{(2\pi D\tau)^{1/2}}{\xi_c}\exp\left(\frac{\xi_c^2}{2D}\tau\right). \tag{6}$$

TG subsequently proposed a bridging formula involving (6) and the $\tau\to0$ result to represent the mean first-passage time for all τ:

$$T=\exp\left(\frac{V_0}{D}\right)\left[\frac{\pi}{\alpha\sqrt{2}}+\frac{(2\pi D\tau)^{1/2}}{\xi_c}\exp\left(\frac{\xi_c^2}{2D}\tau\right)\right], \tag{7}$$

where $V_0=\alpha^2/4\beta$. According to TG, Eq. (7) then is the mean first-passage time for x to make a transition from one metastable state to the other. The exponential prefactor $\exp(V_0/D)$ is not derived within the fluctuating potential argument but it has been justified on other grounds in Ref. 2. Our purpose here is to discuss the coefficient of τ in the second exponential factor in (7) and not this prefactor. In particular, according to Eq. (7) a plot of $\ln T$ vs τ should yield for large τ essentially a straight line of slope

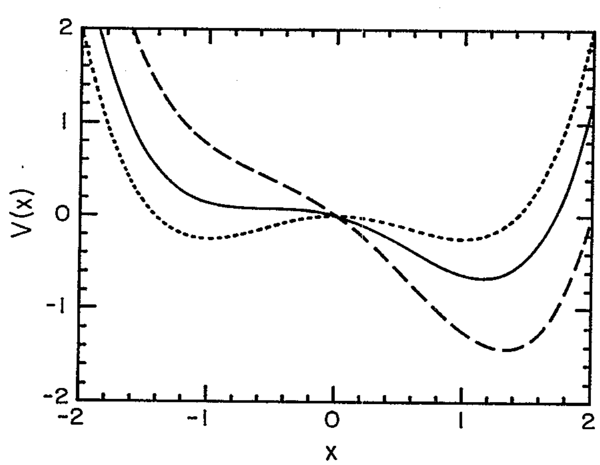

FIG. 1. The "potential" of Eq. (3) with $\alpha=\beta=1$ for three values of ξ. Dashed curve: $\xi>\xi_c$; solid curve: $\xi=\xi_c$; dotted curve: $\xi<\xi_c$.

$\xi_c^2/2D$ (deviations from a straight line predicted by the theory are small but visible as τ decreases).[4,5]

Using a continued fraction method to calculate an eigenvalue closely related to T, Risken and co-workers have shown that for large but finite τ ($\tau<3$) the slope of $\ln T$ vs τ is *greater* than $\xi_c^2/2D$ by a substantial D-dependent amount (approximately 20% for $D=0.2$).[6] Our own (unpublished) numerical simulations of Eq. (1) confirm this result.[7] Herein we provide an explanation of this behavior in the context of the TG model and we give a lower bound for the slope of $\ln T$ vs τ.

Figure 2 shows a typical (single) trajectory of x and ξ for two values of τ, one large but finite ($\tau=5$) and one extremely large ($\tau=50$). In each of these trajectories, the transition from one metastable state to the other is clearly visible and takes place when ξ reaches a value *considerably greater* than ξ_c. In fact, the noise crosses the value ξ_c several times without effect on x.

The reason for this behavior can clearly be seen upon examination of the "potential" curves in Fig. 1. When $\xi=\xi_c$, the potential near the marginally stable state is extremely flat and it takes the process a very long (∞) time T_r to "roll" down to the minimum of the potential. In fact, during this long time, the noise (which is only correlated over a time τ) will change and the previously marginally stable state may eventually become metastable again. We suggest that a successful transition requires that the passage be complete before the noise changes its value, i.e., that

$$T_r \lesssim \tau . \qquad (8)$$

For this condition to be satisfied, ξ must exceed a τ-dependent value $\xi_\tau^0>\xi_c$ given below, a value that increases with decreasing τ.

To estimate the τ dependence of ξ_τ^0, we note that the (deterministic) time for the process to go from a point x_1 to a point x_2 for a fixed ξ (cf. Fig. 3) is given by

$$T_r=\int_{x_1}^{x_2}\frac{dx}{\alpha x-\beta x^3+\xi} , \qquad (9)$$

and it is this time that enters the inequality (8). For the upper limit, we choose a value representative of "arrival at the right-hand well," e.g., $x_2=0$. For the lower limit, we choose $x_2=-(\alpha/3\beta)^{1/2}$, the position of the marginally stable state when $\xi=\xi_c$. Small changes in these choices do not materially affect our estimate of ξ_τ^0 for large τ.

We thus propose the following formula for the mean first-passage time when τ is large but finite:

$$T=\exp\left[\frac{V_0}{D}\right]\left[\frac{\pi}{\alpha\sqrt{2}}+\frac{(2\pi D\tau)^{1/2}}{\xi_\tau}\exp\left[\frac{\xi_\tau^2}{2D}\tau\right]\right] , \qquad (10)$$

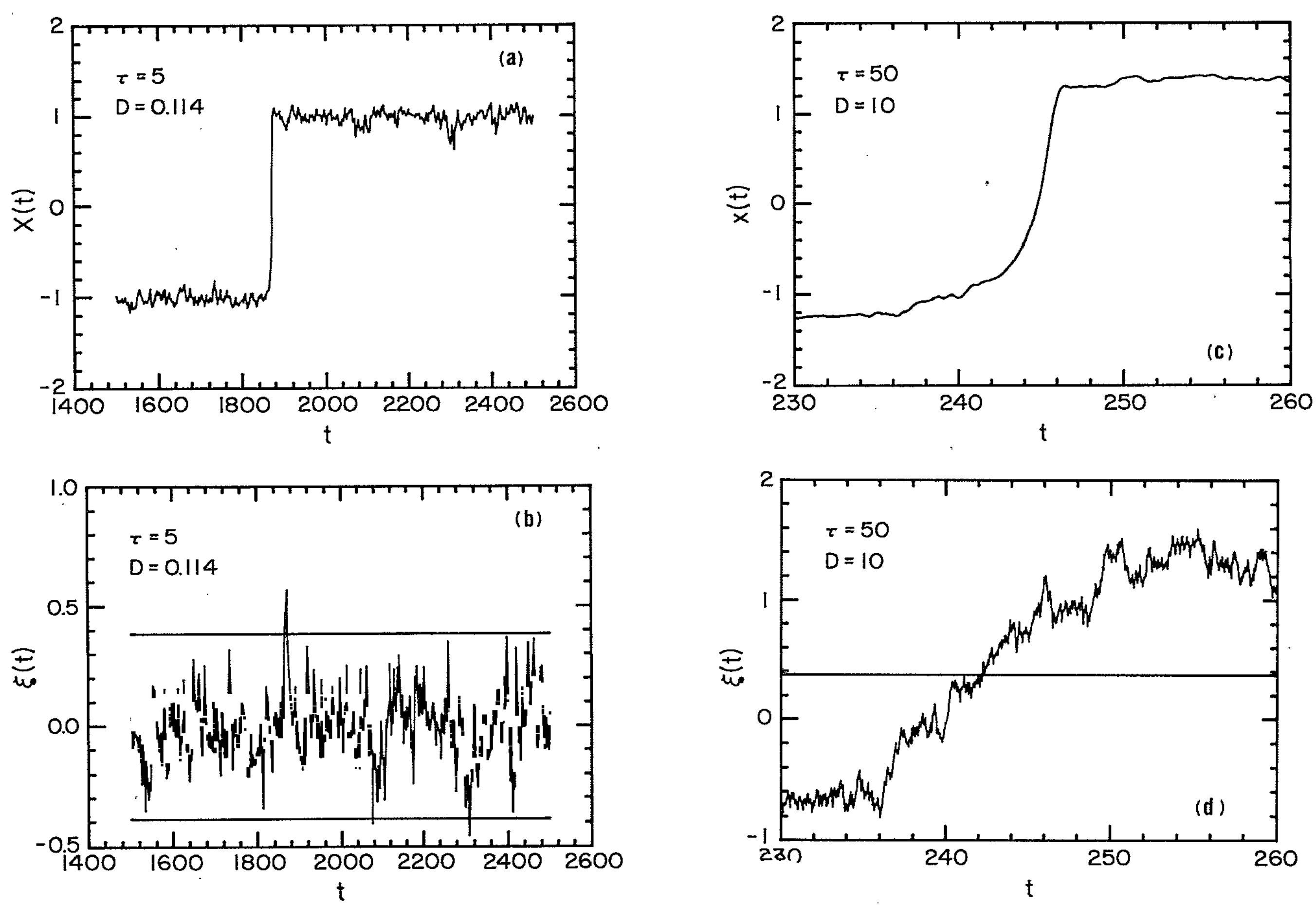

FIG. 2. Typical trajectories generated by Eqs. (1) and (4) with $\alpha=\beta=1$. In (a) and (b) $\tau=5$ and $D=0.114$. In (c) and (d) $\tau=50$ and $D=10$.

where

$$\xi_\tau \geq \xi_\tau^0 \tag{11}$$

and ξ_τ^0 is found from the equality

$$\int_{-1/\sqrt{3}}^{0} \frac{dx}{\alpha x - \beta x^3 + \xi_\tau^0} = \tau . \tag{12}$$

Figure 3 shows ξ_τ^0 vs τ obtained from Eq. (12). Clearly, for the values of τ considered in most available simulations and numerical calculations ($\tau < 5$), the noise indeed has to be considerably larger than ξ_c for passage from one metastable state to another to occur. An analytic expression for this curve can be obtained from the integral (12), which we have performed for large τ:

$$\xi_\tau^0 = \xi_c + \frac{\pi^2}{4\sqrt{3}} \frac{1}{\tau^2} + O(\tau^{-3} \ln \tau) , \tag{13}$$

where we have set $\alpha = \beta = 1$.

Let us consider the specific trajectories shown in Fig. 2 in light of these predictions. One should be aware that single trajectories may behave quite differently from the behavior envisioned when one talks about average results. Thus, it could even happen (although it does not in the trajectories shown here) that a transition occurs when the instantaneous value of the noise is actually smaller than ξ_c. It should also be noted that even a very large value of ξ may not lead to a transition if that value is not retained for a sufficiently long time: after all, the correlation time τ is again only an average. The trajectory for $\tau = 50$ shows a transition when $\xi = 0.48$, i.e., a value above ξ_c. Note that in this particular trajectory an even larger value of ξ attained earlier did not lead to a transition because it was not retained for a sufficiently long time. Our lower-bound estimate indicates that, for such a large $\tau = 50$, the value of the noise required for a successful transition is, on the average, greater than $\xi_{50}^0 = 0.3855$. The $\tau = 5$ trajectory shown in Fig. 2 requires a ξ of 0.57 before the transition occurs. Our lower bound indicates that the value of ξ for a successful transition must, on the average, be greater than 0.4419, again consistent with this result. Numerical simulations[7] yield $T = 8575$ for $D = 0.114$; this value is obtained from Eq. (10) with $\xi_\tau = 0.5024$ (that the particular realization shown in Fig. 2 requires a larger value of ξ is of course not inconsistent with this result).

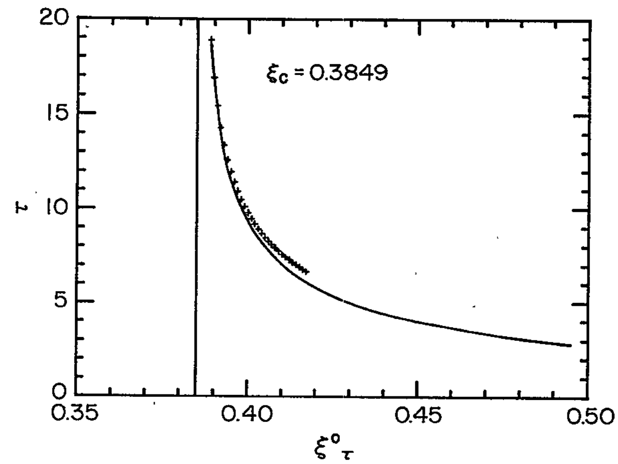

FIG. 3. Numerical solution of Eq. (12) (solid line) and analytic large-τ estimate as given in Eq. (13) (+). The vertical line indicates the critical value ξ_c.

In summary, we have provided an explanation for the deviations of numerical simulations and eigenvalue calculations from the large-τ results of Tsironis and Grigolini. We have provided a quantitative lower bound for the mean first-passage time when the correlation time of the noise is large but finite. We note that the TG theory on which our analysis is based and our arguments to improve it are not related in any way to existing effective Fokker-Planck theories.

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