PHYSICAL REVIEW A VOLUME 40, NUMBER 7 OCTOBER 1, 1989

Insight toward the first-passage time in a bistable potential with highly colored noise

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(Received 30 June 1989)

The exponential coefficient in the first-passage-time problem for a bistable potential with highly colored noise is predicted by be $\frac{8}{27}$ by all existing theories. On the other hand, we show herein that all existing numerical evidence seems to indicate that the coefficient is actually larger by about $\frac{4}{3}$, i.e., that the numerical factor in the exponent is approximately $\frac{32}{81}$. Existing data cover values of $\tau V_0/D$ up to ~ 20, where V_0 is the barrier height, τ the correlation time of the noise, and D the noise intensity. We provide an explanation for the modified coefficient, the explanation also being based on existing numerical simulations. Whether the value $\frac{8}{27}$ predicted by all large-τ theories is achieved for even larger values of $\tau V_0/D$ is unknown but appears questionable (except perhaps for enormously large, experimentally inaccessible values of this factor) in view of currently available results.

A number of theoretical approaches have recently been used to obtain the mean first-passage time from one well to the other of a bistable system driven by highly colored noise.[1-6] The generic system of interest evolves according to the dynamical equation

$$\dot{X}(t) = aX - bX^3 + f(t)\,, \tag{1}$$

and the noise $f(t)$ is assumed to be zero centered and Gaussian with exponential correlations

$$\langle f(t)f(t')\rangle = \frac{D}{\tau}e^{-|t-t'|/\tau}. \tag{2}$$

D is the noise intensity and τ its correlation time. The "potential"

$$V(X) = \frac{b}{4}X^4 - \frac{a}{2}X^2\,, \tag{3}$$

implicit in Eq. (1) has two minima separated by a barrier of height $V_0 = a^2/4b$. The noise $f(t)$ is said to be highly colored when τ/D is large compared to the deterministic quantity V_0^{-1}:

$$\tau V_0/D \gg 1\,. \tag{4}$$

A number of methods have been used to calculate the mean first passage time T from one well of the potential (3) to the other.[7] Although these methods are all approximate and different from one another, there seems to be universal agreement as to the dominant features of the result in the "large-τ" limit $\tau \to \infty$. In this limit [cf. Eq. (4)] all the large-τ theories lead to[1-7]

$$\lim_{\tau\to\infty} T \approx \left(\frac{27\pi D\tau}{8V_0}\right)^{1/2} \exp\left(\frac{8}{27}\frac{V_0\tau}{D}\right). \tag{5}$$

Furthermore, there also seems to be universal agreement that the asymptotic result is achieved very slowly and requires extremely large values of the exponent.[2-5,8]

The approximations that lead to Eq. (5) are typically not systematic expansions in any parameter; consequently, there exists considerable disagreement and uncertainty as to the corrections to this result with decreasing τ, both in the prefactor and in the exponent. It should be stressed that these differences notwithstanding, all the corrections consist of (small) additions to the exponent $8V_0\tau/27D$ (and less importantly and not of particular interest here, small additions to the $\tau^{1/2}$ prefactor). We have also recently developed an argument that leads to such a correction.[6] Our argument is based on the approach of Tsironis and Grigolini,[2] who obtain (5) by assuming that a transition from one well to the other occurs when the noise $f(t)$ reaches the critical value $\mu_c \equiv (16V_0/27)^{1/2}$ at which the "effective potential"

$$V_{\rm eff}(X,f) = V(X) - Xf \tag{6}$$

first ceases to be bistable. We note that this argument is valid only when $\tau \to \infty$, whence $f(t)$ is essentially constant and μ_c marks the separatrix between the two wells of $V_{\rm eff}(X,f)$. With decreasing τ one must consider the variation of $f(t)$ with time, and $f(t)$ must in general reach a value greater than μ_c for a successful transition to occur.[4,6] We do not repeat the detailed argument here,[6] but simply state our result, which replaces Eq. (5) with

$$T=\frac{(2\pi D\tau)^{1/2}}{\mu(\tau)}\exp\left[\frac{\mu^2(\tau)}{2D}\tau\right]. \tag{7}$$

Equation (5) corresponds to the choice $\mu(\tau)=\mu_c$ while we obtain

$$\mu(\tau)=\mu_c(1+1/\tau)=(\tfrac{4}{27})^{1/2}(1+1/\tau). \tag{8}$$

In (8) and frequently henceforth we have set $a=b=1$, so that $V_0=\frac{1}{4}$.

In Ref. 6 we have plotted our simulation results[9] for the mean first passage time in the form $\ln(T/\tau)$ vs $(\tau/D)(1+1/\tau)^2$. Our simulations were carried our for three different values of D in the range $5<(\tau V_0/D)(1+1/\tau)^2<20$, and all our points fall close to the same line on this plot, thus confirming the functional dependence on $\mu(\tau)\tau/D$ predicted in the exponent of (7) (see Fig. 1).

Having confirmed the expected functional dependence in the exponent, our next question (and the central question of this Rapid Communication) is to investigate the value of the slope $8V_0/27=\frac{2}{27}$ predicted in Eqs. (5) and (7) and in all other large-τ theories as well. To explore this question we sought the best fit of our simulations for the parameters α and β arbitrarily introduced as follows:

$$T=\alpha\left[\frac{27\pi D\tau}{2\beta}\right]^{1/2}\frac{1}{(1+1/\tau)}\exp\left[\frac{2}{27}\beta\frac{\tau}{D}\left[1+\frac{1}{\tau}\right]^2\right]. \tag{9}$$

Equation (7) with (8) corresponds to the choice $\alpha=\beta=1$. We find the best agreement between simulations and (9) to occur for the choices $\alpha=6.17$ and $\beta=1.29\approx\frac{4}{3}$. In Fig. 1 we have plotted our simulation results for $\ln(T/\tau)$ for three values of D as a function of $(\tau/D)(1+1/\tau)^2$ (as we did in Ref. 6). The results fall on a straight line of slope $2\beta/27\approx\frac{8}{81}$, indicated as a solid line. On the same graph (arbitrarily starting at the same point on the abscissa simply to highlight the different slopes) we show the dotted line whose slope is $\frac{2}{27}$, representing the large-τ theoretical predictions.

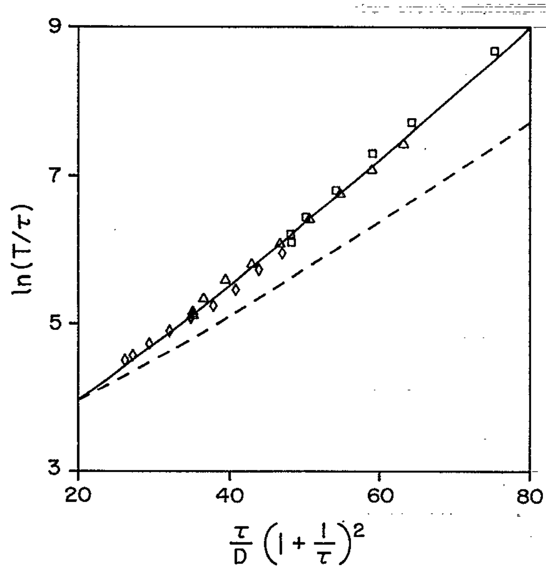

FIG. 1. $\ln(T/\tau)$ vs $(\tau/D)(1+1/\tau)^2$ for $\tau\gtrsim0.9$. The symbols denote numerical simulation results with $D=0.083$ (squares), $D=0.114$ (triangles), and $D=0.153$ (rhombuses). Solid line: Eq. (9) with $\alpha=6.17$ and $\beta=\frac{4}{3}$. Dashed line: the same function with $\beta=1$ and the same abscissa.

Note that the difference in the slopes between the theoretical predictions and the numerical simulations reported in Fig. 1 represents a change in the *coefficient* of the *leading term* in the exponent of T, a coefficient about which no question has been raised heretofore. In view of this unexpected behavior, we returned to some of the existing (far from plentiful) numerical data to be found in the literature and found confirmation of the behavior reported here. Fox[8] reports two large-τ simulations in the range $1<\tau<10$ for $D=0.1$ and $D=0.2$. These values correspond to $\tau V_0/D$ withing the same range of values as we have considered. For $D=0.1$ Fox obtains a slope of 1.0 from his simulations, whereas the large-τ theories would have predicted $8V_0/27D=0.74$. For $D=0.2$ Fox's simulations give a slope of 0.5 instead of the predicted value 0.37. Note that the ratios 0.5/0.27 and 1.0/0.74 are indeed approximately $\frac{4}{3}$. The numerical results of Jung and Hänggi[10,11] for the lowest eigenvalue λ of the two-dimensional Fokker-Planck operator are restricted to the range $\tau V_0/D\leq2.5$ and therefore may not be in the regime appropriate for our large-τ analysis. Nevertheless, we observe that in this case the slope of $\ln(\lambda^{-1}/\tau)$ vs $(\tau V_0/D)(1+1/\tau)^2$ exceeds $\frac{8}{27}$ by an amount not inconsistent with our value of β.

We thus conclude that the existing numerical evidence points to a dependence of T on τ which is exponentially larger than that predicted by the large-τ theories. It may of course be possible that the predicted large-τ result is recovered [i.e., that β in Eq. (9) $\rightarrow 1$] if $\tau V_0/D$ greatly exceeds the values presently accessible to numerical computation. On the other hand, we note that the exponent in Eq. (9) achieves values of $O(10)$ in the available results. The explanation of the observed results, we believe, can be found in the numerical studies of Mannella and Palleschi[5] and their analysis of the results. To describe their analysis, we first note that the large-τ theories are all based on the notion that the process first reaches a particular curve in (x,f) phase space in time T, and that the further passage of the process from that curve to the other well occurs deterministically in essentially zero time. The agreed upon curve that must be reached is the (τ dependent) separatrix between the wells. Since the separatrix cannot be specified analytically for large τ, approximations are made; different theories specify this curve in different ways. In spite of these differences, all the theories lead to the same exponential coefficient for T in the large-τ limit.

The detailed analysis of individual trajectories by Mannella and Palleschi[5] reveals the problem with this viewpoint: In reality there is considerable dynamics in the region of the separatrix, as shown in Figs. 2 and 3. These figures show that once a trajectory reaches the region in (x,f) space from which the theories assume immediate passage to the other well, the actual trajectory spends a

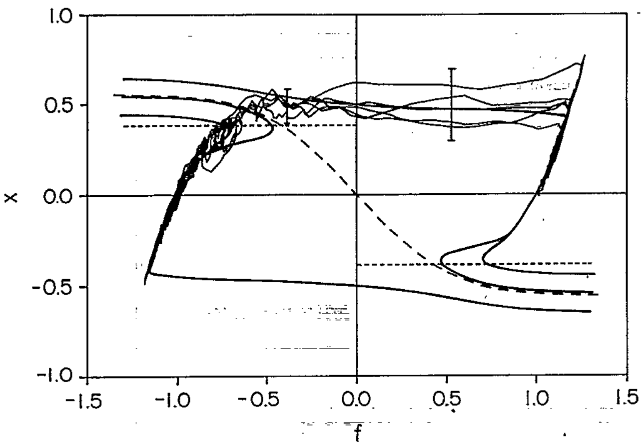

FIG. 2. A number of trajectories in phase space for $D=0.5$ and $\tau=15$. Note the dynamics in the region of the separatrix. The first vertical bar indicates the distribution of values of $f(t)$ in the vicinity of the separatrix. The second vertical bar is twice as long as the first and indicates that even in the process of crossing, the trajectories spread.

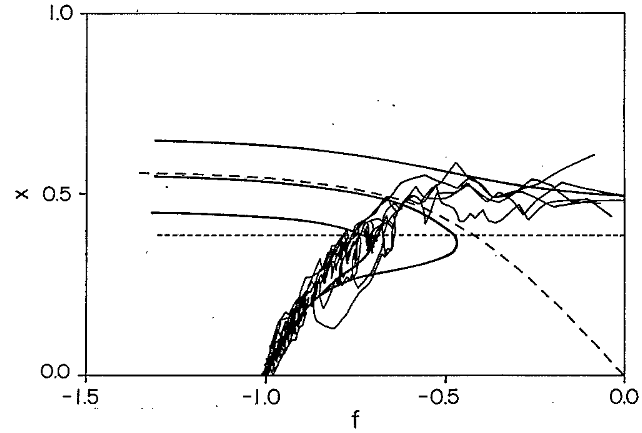

FIG. 3. Expanded view of Fig. 2 in the region of the separatrix.

considerable time crossing and recrossing this region before moving on. This "residence time" contributes exponentially to the mean first passage time.

To estimate the effect of the residence time of the process in the region of the separatrix we follow the reasoning of Mannella and Palleschi and write the mean first passage time $\overline{T}$ as

$$\overline{T}=\int d\mu\, T(\mu)P(\mu)\,. \tag{10}$$

Here $T(\mu)$ is the mean first passage time for $f(t)$ to first reach the value μ and is given by[6,12]

$$T(\mu)=\tau\sqrt{\pi}\{1+\mathrm{erf}[(\tau/2D)^{1/2}\mu]\}e^{(\tau/2D)\mu^2}F[(\tau/2D)^{1/2}\mu] -2\tau\int_0^{(\tau/2D)^{1/2}\mu}dyF(y)\approx\frac{(2\pi D\tau)^{1/2}}{\mu}e^{\mu^2\tau/2D} \tag{11}$$

[cf. Eq. (7)], where $\mathrm{erf}(x)$ is the error function and $F(x)$ is the Dawson integral.[13] The second expression follows from the first when a steepest descent analysis is carried out for large $\mu^2\tau/D$. The long-τ theories identify (11) with the transition time from one well to the other, each theory with a particular choice of μ. $P(\mu)$ in (10) is the probability density for a crossing to the other well to actually occur when $f(t)$ reaches the interval $(\mu,\mu+d\mu)$, and the bar in $\overline{T}$ indicates the average over this distribution. Mannella and Palleschi find the distribution $P(\mu)$ from their simulations. They observe that it is centered at a value that decreases towards μ_c with increasing τ, that it has a width that also decreases with increasing τ, and that it falls off more rapidly than a Gaussian in the wings of the distribution. Their simulations do not reveal whether $P(\mu)$ eventually (i.e., with increasing τ) becomes a δ function at μ_c or whether it retains a finite width. Only if the former occurs would $\overline{T}$ eventually reach the value (5) with the exponential slope $\frac{8}{27}$.

No one has yet developed a theory that leads to an analytic prediction of the distribution $P(\mu)$. We may nevertheless deduce its qualitative effects via phenomenological arguments. For this purpose we choose $P(\mu)$ to be a Gaussian centered at μ_c and of width σ. Since the stochasticity manifest in $P(\mu)$ must of course arise from the random nature of $f(t)$, the width σ must be a function of the parameters of $f(t)$. Thus, it seems to be reasonable to write $\sigma=cD/\tau$ where c is an unknown coefficient whose value we expect to be of $O(1)$. This admittedly *ad hoc* choice for the width is consistent with the numerical results of Mannella and Palleschi and causes $P(\mu)$ to have the observed narrowing tendency as τ increases. Thus we write

$$P(\mu)=(2\pi cD/\tau)^{1/2}e^{-(\mu-\mu_c)^2\tau/cD} \tag{12}$$

[we require that $c<1$; otherwise (10) leads to unphysical results and the distribution (12) must be modified to have a sharper cutoff as observed in the simulations]. The integral (10) with (11) and (12) can be evaluated approximately using a steepest-descent procedure to yield for the mean passage time

$$\overline{T}\approx\frac{(2\pi D\tau)^{1/2}}{\mu_c}(1-c)^{1/2}\exp\left[\frac{\mu_c^2\tau}{2D(1-c)}\right] \approx\left[\frac{27\pi D\tau}{8V_0}(1-c)\right]^{1/2}\exp\left[\frac{8V_0}{27D(1-c)}\tau\right]. \tag{13}$$

The effect of the residence time reflected in $P(\mu)$ is thus to increase the slope in the exponent from $\frac{8}{27}$ to $8/27(1-c)$. The choice $c=\frac{1}{4}$ [which corresponds to $P(\mu)$ having half the width of the Gaussian distribution for $f(t)$ itself] leads to a slope of $\frac{8}{27}\times\frac{4}{3}=\frac{32}{81}$, consistent with our numerical simulations.

In conclusion, we have argued that the finite residence time in the region of the separatrix and the resultant distribution of first passage times leads to exponentially large effects in the mean first passage time for a bistable system driven by highly correlated noise. In all the large-τ simulations carried out to date the exponential correction of the theoretical mean first passage time is approximately $\frac{4}{3}$. Whether the existing large-τ theories correctly predict results for values of τ well beyond those presently accessible remains an open question.

Partial support from the U.S. Department of Energy Grant No. DE-FG03-86ER13606 is gratefully acknowledged by K.L. Three of us (L.R.-P., J.M.S., and F.J.R.) acknowledge financial support from the Dirección General de Investigación Científica y Técnica (Spain, Grants No. 0024/85, No. AE87-0035, and No. PB87-0014). This work was completed while K.L. visited the other authors is Spain. The hospitality of the two institutions is gratefully acknowledged.

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