SOME EXACT RESULTS FOR THE LATTICE COVERING TIME PROBLEM

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ABSTRACT

The lattice covering time of a random walk in finite lattices has recently been defined as the mean time taken by the lattice walker to visit all the sites of the lattice. We solve the lattice covering time problem exactly in one dimension both for reflecting and periodic boundary conditions.

1. Introduction

Recently Nemirovsky et al. [1] introduced the so called lattice covering time problem in lattice random walks. Given a lattice of N sites, the problem is to find the mean time required for the lattice walker to visit all the sites of the lattice. They carried out numerical experiments on a computer to investigate this problem for simple symmetric (nearest-neighbor, unbiased) walks on hypercubic lattices of dimensions D= 1, 2, 3, and 4 with reflecting and periodic boundary conditions. In particular, based on the analysis of small lattices and the statistics of larger ones, they conjectured [1] that the lattice covering time in D=1 is given by the expression

$$t_R(s) = N(N-1) + (s-1)(N-s), \tag{1.1}$$

in the case of reflecting boundary conditions, where s (=1, 2, .., N) is the starting site of the random walk, and

$$t_P = \frac{N(N-1)}{2}, \tag{1.2}$$

in the case of periodic boundary conditions.

In this paper we prove the conjectured formulae (1.1) and (1.2) by solving the lattice covering time problem exactly in one dimension. Our strategy will be to relate the covering time problem to the well known first passage time problem, which can be solved using standard methods [2, 3, 4]. In this respect, we will use the generating function method [4] to calculate the random walk generating function for rings with and without defects that will be related to the generating function for our original lattice with reflecting and periodic boundary conditions, respectively. We will limit our discussion to simple symmetric walks in which at each unit time the walker steps to the right or to the left with equal probabilities. In Sect. 2 and 3 we present the results for the reflecting and periodic boundary conditions cases, respectively. In the Appendix we present the general calculation of the random walk generating function for a ring with and without defects.

2. Reflecting Boundary Conditions Case.

We consider a random walk restricted to the sites $x=1, 2, .., N$ along the x-axis. Reflecting barriers [2] are located at $x= 1/2$ and $x= N + 1/2$. The lattice covering time is defined by the average

$$t_R(s) = \langle t \rangle = \sum_{t=0}^{\infty} t\, f_t^R(s), \tag{2.1}$$

where $f_t^R(s)$ is the probability of all the lattice sites being visited for the first time after t steps by a random walker that had started from the site s.

In the simplest situation in which a walker starts from one of the boundary sites, say the site at $x=1$, it is clear that the lattice covering time is identical to the mean first passage time through the site at $x=N$. To consider the general situation in which the walker starts from any site s the argument is the following: In order for the walker to visit all the sites and end at site N(l) in t steps he must have first visited the site 1(N) at some previous time t' and then went to N(1) after $t-t'$ steps. Then, we can relate the lattice covering probability $f_t^R(s)$ to the more familiar concept of first arrival probability [4]

$$f_t^R(s) = \sum_{t'=0}^{t} \tilde{f}_{t'}^R(1/s) f_{t-t'}^R(N/1) + \sum_{t'=0}^{t} \tilde{f}_{t'}^R(N/s) f_{t-t'}^R(1/N). \tag{2.2}$$

$f_t^R(\ell/s)$ is the probability of the walker arriving at ℓ for the first time after t steps having started from the site s, and $\tilde{f}_t(\ell/s)$ is defined similarly but with the additional restriction that the walker has not visited the boundary sites 1 or N previously. The symmetry of the problem implies that

$$\begin{aligned} f_t^R(1/N) &= f_t^R(N/1), \\ \tilde{f}_t^R(N/s) &= \tilde{f}_t^R(1/N-s+1). \end{aligned} \tag{2.3}$$

Hence

$$f_t^R(s) = \sum_{t'=0}^{t} \left[\tilde{f}_{t'}^R(1/s) + \tilde{f}_{t'}^R(1/N-s+1)\right] f_{t-t'}^R(N/1). \tag{2.4}$$

At this point, we make use of the generating function method [4]. If we introduce the lattice covering generating function

$$F_R(z;s) = \sum_{t=0}^{\infty} z^t f_t^R(s), \tag{2.5}$$

then the lattice covering time (2.1) can be written as

$$t_R(s) = \left.\frac{\partial F_R(z;s)}{\partial z}\right|_{z=1} \tag{2.6}$$

In other words, the lattice covering time is minus the coefficient of the linear term in the representation of the generating function $F_R(z;s)$ as a power series expansion in $1-z$.

Defining the new generating function

$$\tilde{F}_R(z;\ell/s) = \sum_{t=0}^{\infty} z^t f_t^R(\ell/s), \qquad (2.7)$$

and similarly for $\check{F}_R(z;\ell/s)$ in terms of the probabilities $\{\check{f}_t^R(\ell/s)\}$ we obtain

$$F_R(z,s) = \left[\tilde{F}_R(z;1/s) + \check{F}_R(z;1/N-s+1)\right] F_R(z;N/1). \qquad (2.8)$$

Now, the strategy is to relate the generating functions that appears at the right hand site of Eq. (2.8) to the generating functions of periodic lattices or rings with a single defect for which analytical results can be obtained [4]. First, we consider a random walk on a ring of N sites and denote by $p(\ell' \to \ell)$ the probability of the step $\ell' \to \ell$. Then, the probability $P_t(\ell/s)$ that the walker starting from the site s is at site ℓ after t steps satisfies the recursion relation

$$P_t(\ell/s) = \sum_{Q'=0}^{N-1} p(\ell' \to \ell)\, P_{t-1}(\ell'/s), \qquad (2.9)$$

with initial conditions $P_0(\ell/s) = \delta_{\ell s}$

We define the random walk generating function $P(z,\ell/s)$ by

$$P(z;\ell/s) = \sum_{t=0}^{\infty} z^t P_t(\ell/s). \qquad (2.10)$$

The probability $f_t(\ell/s)$ of the walker that started at s arriving at ℓ for the first time after t steps can be related to $P_t(\ell/s)$ in the usual way [4]: In order for the walker to arrive at ℓ after t steps, he must have first arrived at ℓ at some previous time $t'<t$ and then have returned to ℓ after t-t' steps,

$$P_t(\ell/s) = \sum_{t'=0}^{t} f_{t'}(\ell/s)\, P_{t-t'}(\ell/\ell). \tag{2.11}$$

Using Eq. (2.10) we obtain

$$P(z;\ell/s) = F(z;\ell/s)\, P(z;\ell/\ell), \tag{2.12}$$

where the generating function $F(z;\ell/s)$ is defined as

$$F(z,\ell/s) = \sum_{t=0}^{\infty} z^t f_t(\ell/s). \tag{2.13}$$

We note that the expression (2.12) differs, for $\ell=s$, from what we find in Refs. [4] due to our definition of the generating function (2.13) which includes the term $t=0$.

Let us introduce a single defect located, for example, at the origin $\ell=0$. The defect is such that on arriving at it the walker has a probability p of standing still, a probability $q=1-p$ of stepping to the site $\ell=1$, and zero probability of stepping to the site $\ell=N-1$. Now, we are in position to see that for some values of p the corresponding random walk generating functions of a ring with such a defect are equal to the generating functions $\breve{F}_R(z;1/s)$ and $F_F(z;N/1)$ which appear in Eq.(2.8).

Let us consider the case of a trap ($p=1$) located at the origin. Then the probability $f_t(1/s)$ of the walker on the ring that starts from s and reaches $\ell=1$ for the first time without being trapped by the origin is equal to the probability $\breve{f}_t^R(1/s)$ of the walker on our original lattice that starts from s and reaches $\ell=1$ for the first time without visiting site $\ell=N$. This leads to the equality of the generating functions:

$$\tilde{F}_R(z; 1/s) = F(z; 1/s)\Big|_{p=1} . \tag{2.14}$$

Analogously, we may consider the case in which the origin acts as an elastic barrier ($p=1/2$). Then, the probability $f_t(N-1/0)$ of the walker on the ring that starts from $\ell=0$ and reaches $\ell=N-1$ for the first time is clearly equal to the probability $f_t^R(N/1)$ of the walker in our original lattice that starts from $\ell=1$ and reaches $\ell=N$ for the first time. Hence,

$$F_R(z; N/1) = F(z; N-1/0)\Big|_{p=1/2} . \tag{2.15}$$

In the appendix we present the calculation of the random walk generating function $P(z; \ell/s)$ which is related with $F(z; \ell/s)$ by Eq. **(2.12)**. Then, taking into account (2.14), **(2.12)** and the results from the appendix (A.7), (A.8), (A.9), **(A.10)** we obtain

$$\begin{aligned} &\tilde{F}_R(z; 1/s) + \tilde{F}_R(z; 1/N-s+1) = \\ &= 1 - (N-s)(s-1)(1-z) + O((1-z)^2) . \end{aligned} \tag{2.16}$$

In a similar way we have

$$F_R(z; N/1) = 1 - N(N-1)(1-z) + O((1-z)^2). \tag{2.17}$$

Substitution of the results **(2.16)** and **(2.17)** in **(2.8)** gives

$$F_R(z;s) = 1 - \left[N(N-1) + (N-s)(s-1)\right](1-z) + O((1-z)^2). \quad (2.18)$$

Then, applying (2.6), we arrive at the desired expression (1.1) for the lattice covering time for the lattice with periodic boundary conditions.

3. Periodic boundary conditions case.

Let $t_P(N)$ denote the lattice covering time for a ring of N sites. In order for the walker to cover N sites he should first cover N-1 sites and then proceed to cover the last site. But, the average time required to cover N-1 sites is identical to the lattice covering time for a ring of N-1 sites. Therefore, as illustrated in Fig. 1, we have

$$t_P(N) = t_P(N-1) + \bar{t}, \quad (3.1)$$

where $\bar{t}$ is the average time required to cover the last site. It is clear, however, that $\bar{t}$ is identical to the mean first passage time through the site N of a walker that started from the site 1 on a ring of N sites (Fig. 1) This quantity can be obtained using the results of the appendix. In general, the mean first passage time that the walker, starting from site s, is at site ℓ after t steps is given by

$$\bar{t}_\ell(s) = \sum_{t=0}^{\infty} t\, f_t(\ell/s). \quad (3.2)$$

In the same way as in the previous calculation

$$\bar{t}_\ell(s) = \frac{\partial}{\partial z} F(z;\ell/s)\Big|_{z=1}, \quad (3.3)$$

where $F(z;\ell/s)$ was defined by **(2.13).** In this case we consider the problem without defects. From **(2.12)** and **(A.10)** we obtain

$$\bar{t}_{\ell}(s) = (\ell-s)\,(N-(\ell-s)). \tag{3.4}$$

Then

$$\bar{t} \equiv \bar{t}_{N}(1) = N-1. \tag{3.5}$$

Solving the recursion relation

$$t_P(N) = t_P(N-1) + N - 1, \tag{3.6}$$

with $t_P(1)=0$ we arrive at the result **(1.2)**

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Appendix

The random walk generating function, defined by Eq. **(2.10),** satisfies the following equation

$$P(z;\ell/s) - z\sum_{\ell'=0}^{N-4} p(\ell'\to\ell)\,P(z;\ell'/s) = \delta_{\ell s}. \tag{A.1}$$

In the presence of few defect points the probability of the step $\ell' \to \ell$ can be written in the form

$$p(\ell' \to \ell) = p_0(\ell' \to \ell) + q(\ell' \to \ell), \tag{A.2}$$

where $p_0(\ell' \to \ell)$ corresponds to the case of a simple symmetric random walk without defects for which the probability of the step $\ell' \to \ell$ can be written in the form

$$p_0(\ell' \to \ell) = \frac{1}{2}\,\delta(\ell - \ell' + 1) + \frac{1}{2}\,\delta(\ell - \ell' - 1), \tag{A.3}$$

and $\delta(\ell) = \sum_{K=-\infty}^{\infty} \delta_{\ell+KN,0}$ is the sum over Kronecker's delta. $q(\ell' \to \ell)$ is the perturbation from the simple symmetric walk due to defects.

Inserting **(A.2)** in **(A.1)** and resolving the equation we obtain

$$P(z;\ell/s) = P_0(z;\ell - s) + z\sum_{\ell'=0}^{N-1}\sum_{\ell''=0}^{N-1} P(z;\ell''/s)\, q(\ell'' \to \ell)\, P_0(z;\ell - \ell'), \tag{A.4}$$

where $P_0(z;\ell)$ corresponds to the solution of **(A.1)** and **(A.2)** without defects ($q(\ell' \to \ell) = 0$)

$$P_0(z;\ell) = \frac{1}{\sqrt{1-z^2}}\;\frac{x^{\ell} + x^{N-\ell}}{1 - x^N} \tag{A.5}$$

for $0 \le \ell \le N$, and $x = \dfrac{1-\sqrt{1-z^2}}{2}$. To model the defect described in Sect.2 the perturbation $q(\ell' \to \ell)$ is given by

$$q(\ell' \to \ell) = \delta(\ell')\left[p\,\delta(\ell) + \left[\frac{1}{2} - p\right]\delta(\ell - 1) - \frac{1}{2}\,\delta(\ell + 1)\right]. \tag{A.6}$$

Using this expression for the perturbation the Eq. **(A.4)** becomes

$$P(z;\ell/s) = P_0(z;\ell-s) + z\,P(z;0/s)\Big\{p\,P_0(z;\ell) + \left[\frac{1}{2}-p\right]P_0(z;\ell-1) - \frac{1}{2}P_0(z;\ell+1)\Big\}. \quad \text{(A.7)}$$

Setting ℓ =0 we obtain

$$P(z;0/s) = \frac{P_0(z;s)}{q + p(1-z)P_0(z;0)} \quad \text{(A.8)}$$

Finally, it is useful to give the expansions of the generating function (A.5) in powers of 1-z

$$P_0(z;0) = \frac{1}{N(1-z)}\left[1 + \frac{1}{6}(N^2-1)(1-z) + O((1-z)^2)\right], \quad \text{(A.9)}$$

$$\frac{P_0(z;\ell)}{P_0(z;0)} = 1 - \ell(N-\ell)(1-z) + \frac{1}{6}\ell(N-\ell)\left[N^2-5+\ell(N-\ell)\right](1-z)^2 + O((1-z)^3). \quad \text{(A.10)}$$

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Figure Captions

Fig. 1. The lattice covering time for a ring of N sites is equal to the lattice covering time for a ring of N-1 sites plus the time required to cover the last site.

Fig. 1