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Noise-sustained signal propagation

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Abstract. – A simple model of a stochastic bistable medium is analytically shown to exhibit pulse propagation sustained by external spatiotemporal noise. In particular, signal transmission is seen to be optimal for a certain nonzero value of the noise intensity, which can be determined theoretically. Numerical simulations confirm the analytical predictions. This property of noise enables the use of the bistable medium as an information channel through which a continuous (in general nonperiodic) string of bits can be transmitted.

Bistable systems are very sensitive to the influence of random fluctuations, which can induce transitions between the two stable states of the system in a straightforward way. In the presence of an external sinusoidal signal, bistable systems readily exhibit *stochastic resonance* (SR), a phenomenon according to which the harmonic response of the system to the external signal is enhanced for intermediate noise levels [1]. Recent years have witnessed an increasing interest on the extension of this and other noise-related behaviors to spatially extended systems (see [2] for a recent review). In this sense, spatiotemporal bistable *media* have been reported to display, among others, phenomena such as noise-induced phase transitions [3], noise-induced front propagation [4], array enhanced SR [5], SR of front motion in inhomogeneous media [6], and solitonic SR [7].

Extending the above-mentioned scenario, a question that has been recently addressed, concerning the interplay between noise and spatial coupling in bistable media, is that of signal propagation through a chain of bistable elements. The ultimate purpose of these investigations is to analyze the potentially constructive role of noise on the transmission of information in various kinds of systems, including chemical samples, neural tissue and optical communication devices. Recent experimental results have been reported by Löcher *et al.* [8], who have shown that noise is able to sustain the propagation of a front in a chain of coupled bistable diode resonators. In their experiment, all elements in the chain are initially taken to a *metastable* state by means of a global bias. When the element at one end of the chain is driven to the lower-energy *stable* state, spatial coupling induces propagation of the stable phase into the metastable one in the form of a kink. This propagation can only be observed, in the noise-free case, for a large enough bias. However, spatiotemporal noise is seen to sustain kink propagation even for a subthreshold bias. This system only supports propagation of fronts, and hence it is not suitable for the *continuous* transmission of information. A further step in that direction has been taken by forcing *harmonically* the first element in a chain of oscillators with two *equally stable* states. Two different models have been numerically analyzed, depending on the type of coupling between the elements: one-way (*i.e.*, an element influences only one of its neighbors) [9] and two-way (*i.e.*, an element affects equally its two neighbors) [10]. The first case corresponds to having both diffusive and advective coupling in a continuum description, and the second to having only diffusion. In both situations, an optimal amount of noise enhances the propagation of the harmonic signal down the chain of oscillators. Again, we are still far from the regime of information transmission, because these results are restricted to harmonic signals, whereas any piece of information would generally consist of a nonperiodic string of pulses. Furthermore, no theoretical explanation of those observations has been provided so far.

The present letter aims to find an analytical understanding of noise-sustained pulse propagation in spatially extended systems. In contrast to the above-mentioned studies, where noise had an additive nature, we consider in what follows the influence of external *multiplicative* noise, following the trend of recent experimental results in both subexcitable [11, 12] and excitable [13] chemical media. Other constructive effects of fluctuations recently studied in this kind of systems include incoherent propagation of pulses sustained by noise, which has been numerically modelled in terms of stochastic cellular schemes [14], and has been invoked to explain the so-called "thermal patterns" observed for propagating intercellular calcium waves [15]. Here we will use, instead of an excitable system, a more standard and better controlled model of a stochastic *bistable* medium, suitable for a theoretical analysis of the influence of noise on signal propagation. A bistable medium with two equally stable states may support static kinks linking the two states. A kink-antikink combination with a suitable width corresponds to a stable pulse, which can be moved by drift forces in order to use the system as an information channel. However, a pulse can only maintain its shape in this model as long as the two states of the system are equally stable. When this is not the case, the most stable phase invades the metastable one and the pulse shrinks (or widens) during propagation, eventually disappearing. However, as we will see in what follows, a spatiotemporal multiplicative noise is able to sustain a pulse in that situation, provided the noise intensity belongs to an intermediate range of values. This property is further used to demonstrate the feasibility of this system to transmit a continuous, aperiodic train of information bits.

Let us consider the following stochastic reaction-diffusion-advection model:

$$\frac{\partial \phi}{\partial t} = \phi(1-\phi)(\phi-a) + D\frac{\partial^2 \phi}{\partial x^2} + v\frac{\partial \phi}{\partial x} + \phi\, \xi(x,t), \tag{1}$$

written for a scalar field $\phi(x,t)$ defined in a one-dimensional space. Spatial coupling is here given by the diffusion coefficient D and the advective velocity v. As mentioned earlier, the combination of these two mechanisms (diffusion and advection) corresponds to unidirectional coupling in a discrete chain of oscillators (which could be easily implemented, for instance, in an electronic system).

The field $\xi(x,t)$ appearing in (1) represents an external multiplicative noise that we take to be Gaussian and white in time, with zero mean and correlation

$$\langle \xi(x,t)\,\xi(x',t')\rangle = 2\epsilon C(x-x')\delta(t-t'), \tag{2}$$

defined in terms of a spatial correlation function $C(x)$ to be specified later on. Equation (1) will be interpreted in the Stratonovich sense.

The nonlinear kinetic part of model (1) can be understood in terms of the local bistable potential

$$V(\phi,a) = \frac{a}{2}\phi^2 - \frac{1+a}{3}\phi^3 + \frac{1}{4}\phi^4. \tag{3}$$

For $0 < a < 1$, this potential exhibits two stable fixed points at $\phi = 0$ and $\phi = 1$. The relative stability of these two states is controlled by the control parameter a, since

$$\Delta V(a) = V(1,a) - V(0,a) = \frac{2a-1}{12}\,. \tag{4}$$

From this expression one can see that, for $a = 0.5$, both states are equally stable, whereas for $a > 0.5$, the state $\phi = 0$ is the most stable one. Hence, according to the reasoning presented in the introductory paragraphs, in the absence of noise, stable kink-antikink pulses will only be propagated (advected) for $a = 0.5$: for any other value of $a > 0.5$, $\phi = 0$ and $\phi = 1$ are not equally stable, and thus the pulse becomes unstable and deforms quickly as it propagates.

We will now show that the external noise defined by (2) influences constructively the spatiotemporal propagating behavior of model (1), by allowing virtually stable pulses to exist for any value of $0.5 < a < 1$, provided the noise intensity is appropriately tuned. In order to prove this assertion, we note that, in the Stratonovich interpretation, the multiplicative-noise term in eq. (1) has a nonzero mean value, which can be evaluated by means of standard stochastic calculus [2], as $\langle \phi\,\xi(x,t)\rangle = \epsilon C(0)\langle\phi\rangle$. We can explicitly extract this systematic contribution of the external noise by defining a new stochastic field with zero mean, $\eta(x,t) = \phi\xi(x,t) - \epsilon C(0)\phi$. The resulting effective model then reads

$$\frac{\partial\phi}{\partial t} = \phi(1-\phi)(\phi-a) + \epsilon C(0)\phi + D\frac{\partial^2\phi}{\partial x^2} + v\frac{\partial\phi}{\partial x} + \eta(x,t)\,. \tag{5}$$

The extra term in this equation in turn renormalizes the bistable potential in the following way:

$$V_{\text{eff}}(\phi,a,\epsilon) = \frac{a-\epsilon C(0)}{2}\phi^2 - \frac{1+a}{3}\phi^3 + \frac{1}{4}\phi^4, \tag{6}$$

modifying correspondingly the relative stability of the unchanged stable state $\phi = 0$ and the corrected one $\phi \sim 1 + \epsilon C(0)/(1-a)$. Given this systematic effect, it is readily possible to determine *analytically* the optimal value of the noise intensity for which these two steady states regain equal stability. To first order in ϵ one obtains

$$\epsilon_{\text{op}} = \frac{2a-1}{6C(0)}\,. \tag{7}$$

Hence, for $a > 0.5$, an external noise, whose optimal intensity is given by (7), is able to sustain pulse propagation in a detectable way. In order to check this analytical result, we have performed numerical simulations of model (1) in a one-dimensional space. A narrow pulse is introduced at one end of the system by applying the following boundary condition:

$$\phi(0,t) = \left\{\left[1+\exp\left[-\frac{t-t_1}{\sqrt{\Delta}}\right]\right]\left[1+\exp\left[\frac{t-t_2}{\sqrt{\Delta}}\right]\right]\right\}^{-1}, \tag{8}$$

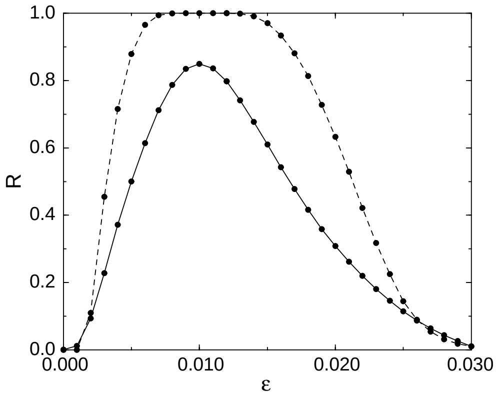

Fig. 1

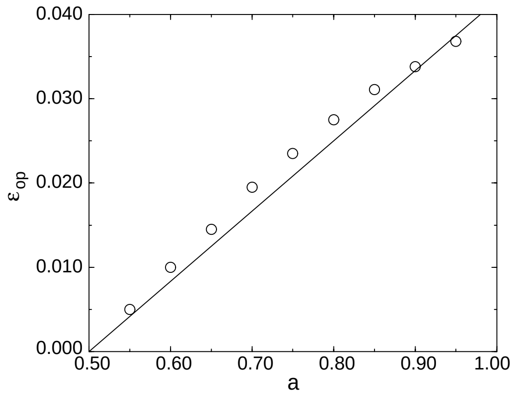

Fig. 2

Fig. 1 – Propagation efficiency rate $\langle R \rangle$, defined by eq. (9), for $a = 0.6$ and increasing noise intensities (solid line). The dashed line is the probability that R reaches a value ≥ 0.25.

Fig. 2 – Optimal noise intensity obtained as the maxima of the propagation efficiency rate R (empty circles) and from the analytical prediction (7) (solid line).

for $t \in (0, T)$, and where $t_1 = (T - \delta_t)/2$, $t_2 = (T + \delta_t)/2$, δ_t being the temporal width of the pulse and T the time interval within which the pulse is confined.

Equation (1) has been numerically integrated using a Heun method [2] in a discrete one-dimensional lattice with $N = 420$ cells of size $\Delta x = 0.25$. An integration time step $\Delta t = 0.01$ has been used. The discrete spatial correlation of the noise C_{ij} has been assumed to be of the form $C_{ij} = (1/\Delta x)\delta_{ij}$, which corresponds in the continuum limit to a spatial white noise. The values of δ_t and T have been chosen in such a way that, in noise-free condition and for $a = 0.5$, pulses do not decay significantly during their propagation: $\delta_t = 5$ and $T = 9$. Other parameters are $\Delta = 0.2$, $v = 2$, and $D = 0.4$.

A single pulse is introduced at one end of the system by means of (8). In order to characterize the efficiency of propagation, we define the integrated transmitted amplitude that reaches the control point at $x = L = 100$, $A = \int_0^T \phi(L, t) \mathrm{d}t$. For $a = 0.5$ and in the absence of noise, the pulse propagates virtually unchanged and a nonzero value of A is obtained ($A = A_0 \simeq 5$ for the parameters chosen). For $a > 0.5$, a nonzero noise intensity is needed in order to sustain propagation. The efficiency of the system can be measured by defining the propagation efficiency rate as

$$R = 1 - \frac{|A - A_0|}{A_0}, \tag{9}$$

where the deterministic case with $a = 0.5$ is used as a reference value. Figure 1 shows the value of $\langle R \rangle$ averaged over 500 realizations of the noise, computed numerically for increasing noise intensities and $a = 0.6$. This quantity exhibits a maximum for an intermediate value of the noise intensity. For smaller noise levels, the state $\phi = 0$ is more stable than $\phi = 1$, and hence the pulse shrinks with propagation, decaying before reaching the final end of the system. For large noise levels, on the other hand, the state $\phi = 1$ is the most stable one, so that the pulse widens and the propagation efficiency, as defined by (9) breaks down, as well.

The value of ϵ for which $\langle R \rangle$ is maximal should correspond, at least approximately, to

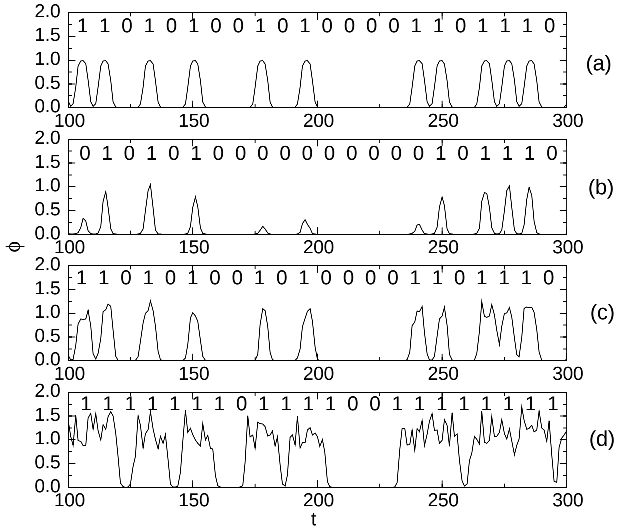

Fig. 3 – Sequence of 22 pulses recorded after reaching position $x = 100$, for $a = 0.5$ and $\epsilon = 0$ (a), and for $a = 0.6$ with $\epsilon = 0.004$ (b), 0.01 (c) and 0.04 (d). The labels on top of each graph denote the corresponding detected bit trains, and in particular that of (a) coincides with the input message.

the optimal value ϵ_{op} predicted above (eq. (7)). A comparison between these two quantities is shown in fig. 2, plotted against the control parameter a. In this figure, the solid line represents the analytical result, and the symbols the corresponding measurements obtained from simulations.

Once we have characterized the ability of the external noise to sustain propagation of a single pulse through the system, we now aim to examine the performance of the model as an information transmitter. To that end, we arbitrarily assume that pulses reaching the final end of the system will be detected provided their integrated amplitude A exceeds at least a 25% of the reference value A_0, *i.e.*, if $A \geq A_0/4$. The probability that R exceeds the

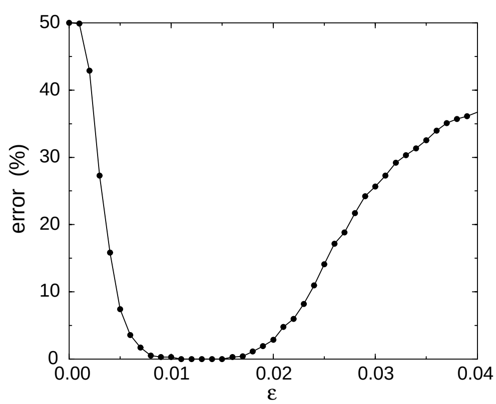

Fig. 4 – Error percentage in the propagation of a message of 1000 bits for $a = 0.6$ and increasing values of the noise intensity.

corresponding threshold $R_{\rm th} = 0.25$ in the previously described 500 realizations is also plotted in fig. 1 as a dashed line. This probability has been estimated from numerical data of $\langle R \rangle$ and $\langle R^2 \rangle$ assuming a Gaussian distribution for R. One can see that, for a nonvanishing range of values around the optimal noise intensity, the pulse will be detected with probability 1. This result indicates that the proposed system might be efficiently used as information channel. To analyze this possibility, we generate, at $x = 0$, a train of N bits in boxes of width T: an empty box ($\phi = 0$) corresponds to a "0" bit, whereas a pulse created by eq. (8) corresponds to a "1" bit. At the final end of the system, the contents of a box are interpreted as a "1" if the detected integrated amplitude within the box exceeds $A_0/4$ (following the arbitrary criterion introduced above for a single pulse), and as a "0" otherwise. The signal reaching the final end of the system and the corresponding detected bit train are shown in fig. 3 for four different situations with the same input signal.

Figure 3(a) corresponds to the reference case, given by the deterministic situation ($\epsilon = 0$) with $a = 0.5$. In this case, the message is transmitted without errors. When $a = 0.6$ and the noise level is small (fig. 3(b)), some pulses reach the final end of the system, but most of them cannot be detected and the transmission efficiency is poor. For intermediate values of the noise intensity in the vicinity of the optimal value (7), virtually all the initial "1" bits reach the detector with a large enough integrated amplitude and are detected. In this regime, shown in fig. 3(c), signal transmission is optimal. Finally, for large noise levels (fig. 3(d)), the prescribed pulses start to widen during propagation, with the result that original "0" bits are detected as "1" bits and the message is again degraded. In summary, errors in the message transmission arise either from shrinking or from widening of the information-carrying bits, with the first mechanism acting for small noise levels, and the second one for large noise levels. These two mechanisms had been commented previously in relation to the propagation of solitary pulses.

Finally, we aim to quantify the efficiency of the communication channel. Several measures have been recently used to characterize information propagation and/or encoding in stochastic dynamical systems; such measures include, among others, mutual information [16] and correlation functions [17]. In this case, however, a direct comparison between the input and output bit strings gives a reasonable quantification of the validity of the transmission process. The result of this comparison, in the form of the percentage of errors in the string, is shown in fig. 4 for increasing values of the noise intensity. In this case a string of 1000 bits has been transmitted through the system, 50% of which are "1" bits randomly distributed along the string. It can be observed that, in the vicinity of the optimal noise level (and in very good agreement with the dashed curve of fig. 1) the error percentage drops virtually to zero.

In conclusion, we have shown that external multiplicative noise is able to sustain pulse propagation in bistable media, for a nonvanishing range of noise levels. The optimal noise intensity can be evaluated analytically in an approximate way, and numerical simulations confirm this result rather satisfactorily. A direct extension of this property is the ability of the system to transmit a continuous and aperiodic train of information bits. An application of it, in a chemical excitable system, is in progress.

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