PHYSICAL REVIEW LETTERS

VOLUME 76 | 22 APRIL 1996 | NUMBER 17

External Fluctuations in Front Propagation

J. Armero,[1] J. M. Sancho,[1,4] J. Casademunt,[1] A. M. Lacasta,[2] L. Ramírez-Piscina,[2] and F. Sagués[3]

[1]*Departament d'Estructura i Constituents de la Matèria, Universitat de Barcelona, Av. Diagonal 647, E-080028 Barcelona, Spain*

[2]*Departament de Física Aplicada, Universitat Politècnica de Catalunya, Av. Dr. Gregorio Marañón 50, E-08028 Barcelona, Spain*

[3]*Departament de Qumíca Física, Universitat de Barcelona, Av. Diagonal 647, E-08028 Barcelona, Spain*

[4]*Institute for Nonlinear Science, Department 0407, University of California, San Diego, 9500 Gilman Drive, La Jolla, California 92093-0407*

(Received 5 December 1995)

We study the effects of external noise in a one-dimensional model of front propagation. Noise is introduced through the fluctuations of a control parameter leading to a multiplicative stochastic partial differential equation. Analytical and numerical results for the front shape and velocity are presented. The linear-marginal-stability theory is found to increase its range of validity in the presence of external noise. As a consequence noise can stabilize fronts not allowed by the deterministic equation. [S0031-9007(96)00015-4]

PACS numbers: 03.40.Kf, 05.40.+j, 47.20.Ky, 47.54.+r

The problem of front propagation has been receiving a great deal of attention in recent years due to its relevance to a large variety of systems in nonlinear physics, chemistry, and biology [1]. Here we will focus on the simplest case in which a globally stable state invades an unstable or metastable state. This problem has been extensively studied in the recent literature [2–8] particularly concerning the issue of velocity selection.

On the other hand, in the last few years there has been a growing interest in the theoretical study of the role of fluctuations in front propagation [7,9–14], and in particular there have been some experiments on the effects of stochastic turbulence in front propagation in the context of chemical fronts [15]. These studies have been basically concerned with the modification of the front velocity and the spreading of the front due to fluctuations.

Internal [9–12] and external [13,14] fluctuations have been introduced in particular models using both Langevin [9,11,13,14] and master equation formalisms [10,12], but no systematic studies have been carried out concerning the modification of the well established selection criteria of the deterministic case. For internal fluctuations mostly numerical studies of different situations have obtained distinct effects on the front propagation. The case with the most direct comparison with the present work [9] found no change in the front velocity. On the other hand, previous analytical approaches for external fluctuations [13,14] have been based on small noise perturbative expansions which turn out to have a rather small range of validity for our purposes.

Here we will introduce a new approach which relies on a physically intuitive picture of the problem but which is nonperturbative. As the accompanying numerical simulations will show, our theoretical approach gives an accurate quantitative description for a very broad range of noise intensities and allows for a general discussion of selection criteria in the presence of external fluctuations.

We focus our study on the simplest prototypical equation for front propagation dynamics, and we introduce fluctuations via a Langevin equation. In our study, noise is assumed to be of external origin and is thus introduced as a stochastic spatiotemporal variation of a control parameter. For example, in an experimental situation such as a nematic liquid crystal in the presence of a magnetic field [8], the control parameter could be expressed in terms of the intensity of this field. This would give rise to a Langevin equation with multiplicative noise, which is the situation we address here.

An additive noise source could also be considered in principle in this problem to model fluctuations from other origins, for instance, internal noise [9]. However, this does not modify the front velocity for the invasion of either metastable or unstable states. Moreover, in the latter case an additive noise would trigger the formation of domains of the two stable phases, modifying the nature of the problem into a phase separation process with diffusive dynamics [9]. Actually, in this case, the front itself does exist only during a short transient. Hence we will omit additive noise sources in the present study.

Our starting point is an equation of the form

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \psi(1 - \psi)(a + \psi), \tag{1}$$

where the coefficient of the linear term, a, stands for an external control parameter. The model has three homogeneous steady states $\psi = 0, 1, -a$. Their stability depends on the particular value of a, which is allowed to vary in the interval $(-\frac{1}{2}, 1)$ to ensure the global stability of the $\psi = 1$ state.

We are interested in the case in which a uniformly propagating front is moving to the right replacing the $\psi = 0$ state by the $\psi = 1$ state. In the absence of noise, this situation has been extensively studied in the last few years [2–5], resulting in the following scenario.

When the $\psi = 0$ state is metastable $(-\frac{1}{2} \leq a < 0)$, the model has a unique front solution with a kinklike profile and a velocity $v_{\mathrm{nl}} = (2a + 1)/\sqrt{2}$.

When $\psi = 0$ is unstable $(0 < a \leq 1)$, there is a continuous degeneracy of solutions for steadily propagating fronts, with a corresponding continuum of possible velocities. All these solutions exhibit an exponential decay e^{-kx}, $x \to \infty$, with different velocities associated with different k's.

For $\frac{1}{2} \leq a \leq 1$, the linear-marginal-stability criterion applies [4,5] and sufficiently localized initial conditions (with a compact support) evolve towards the solution of minimum velocity $v_l = 2\sqrt{a}$ with $k = k_l = \sqrt{a}$. This kind of behavior will be referred to as the linear regime of the model because only the linear part of Eq. (1) is required to predict its velocity. However, for $0 \leq a \leq \frac{1}{2}$, this linear criterion fails and the solution of the full nonlinear Eq. (1) is necessary; hence the name nonlinear regime. In the latter case, initial conditions with $k \geq k^* = \sqrt{2}\,a$ propagate, after a short transient, with the nonlinear velocity v_{nl} and with $k_{\mathrm{nl}} = 1/\sqrt{2}$. This behavior is actually the extrapolation of that of the metastable regime. In fact, k_{nl} corresponds to the invasion mode of Ref. [5], which destabilizes profiles with $k^* \leq k \leq k_l$.

In both linear and nonlinear regimes, higher propagation velocities may be obtained by preparing special initial conditions with $k < k_l$ or $k < k^*$, respectively. In these cases, the front profile keeps its initial asymptotic exponential decay e^{-kx} and it propagates with velocity $v = \frac{k^2 + a}{k}$. These front solutions will be termed hereafter "quenched solutions."

Fluctuations of external origin are introduced by replacing the control parameter a in Eq. (1) by $a \to a(x,t) = a + \xi(x,t)$, with $\xi(x,t)$ a Gaussian noise of zero mean and correlation given by $\langle \xi(x,t)\xi(x',t')\rangle = 2\delta(t - t')\varepsilon(\frac{|x-x'|}{\lambda})$. The function $\varepsilon(x)$ accounts for the spatial correlations of the external fluctuation and the parameter λ is the associated correlation length. We will take λ as finite but much smaller than any other spatial scale, in particular much smaller than the front width. This limiting case could not be adequate for some realistic experimental situations. It would be interesting to test the validity of our predictions as a function of the noise correlation length, but such a study goes beyond the scope of this Letter and is deferred to future work [16].

Incorporating the fluctuations in this way, then Eq. (1) transforms into a stochastic partial differential equation for the evolution of the front in the presence of multiplicative noise

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \psi(1 - \psi)(a + \psi) + \psi(1 - \psi)\xi(x,t). \tag{2}$$

It is worth remarking here that the way control parameter fluctuations appear is such that it preserves the stationary states $\psi = 0$ and $\psi = 1$ connected by the front. In this way the noise is most important at the front region and vanishes at the asymptotic states.

The physical picture on which we base our approximation scheme follows from illustrative simulations of the Langevin Eq. (2) such as Fig. 1. We have used a standard finite-difference Euler algorithm with spatial mesh size Δx and a time step Δt adequate to ensure stability and accuracy. Noise values are taken as independent for different discretization cells, which in practice corresponds to choosing a correlation length λ of order Δx. Integration of the resulting stochastic differential equation is then implemented by means of a standard procedure [17].

As shown in Fig. 1, for moderate noise intensities one sees that the front has a rather well defined position and width, and a basic kinklike shape which is not destroyed by the noise. The position of the front can thus be defined by the integral $z(t) = \int_L dx\, \psi(x,t)$. The instantaneous front velocity for a particular realization is then obtained

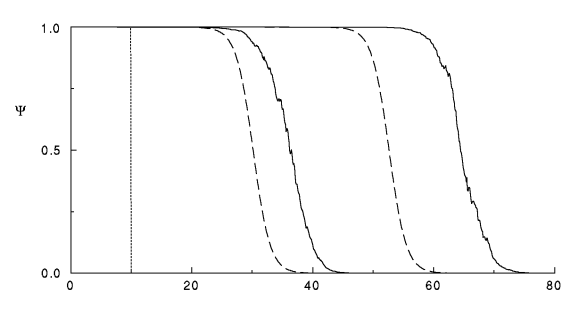

FIG. 1. Two stages of a deterministic front (dashed line) and a fluctuating front (solid line) from a steplike initial profile (dotted line) with $a = 0.3$ and $\varepsilon(0) = 0.3$ ($\Delta x = 0.1$ and $\Delta t = 10^{-3}$).

as $v = \dot{z}$, averaged over an appropriate time window. In Fig. 1 it is also apparent that the noisy front is faster.

Further analysis of simulation data reveals that the noise not only induces a shift of the ensemble averaged velocity but also a diffusive spreading of the front position [18]. In these circumstances the ensemble average $\langle\psi(x,t)\rangle$ exhibits a width which grows in time as $t^{1/2}$ (in the frame moving with the average velocity) [12], but which is not the actual front width. In fact, the front has a well defined mean shape which is different from the deterministic kink solution and which is obtained when the roughness of the front profile produced by noise is appropriately averaged out. These small fluctuations of the front shape relax in a much faster time scale than the wandering of the front position. The separation of time scales of the different effects of the noise is at the heart of our approximation scheme.

In this spirit, and as long as the front has a well defined position and shape, we may proceed formally as follows. We first define an instantaneous distance to the ensemble averaged position as $\Delta(t) = z(t) - \langle z(t)\rangle$, and a displaced profile as $p(x,t) = \psi(x + \Delta(t), t)$.

The equation for $p(x,t)$ is then

$$\frac{\partial p}{\partial t} = \dot{\Delta}\,\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial x^2} + p(1-p)(a+p) + p(1-p)\xi\,. \tag{3}$$

The mean front shape is now given by the ensemble average $p_0(x,t) = \langle p(x,t)\rangle$. Taking the ensemble average of Eq. (3) and using Novikov's theorem [19] for the noise term, we get an equation of motion for $\langle p(x,t)\rangle$. Higher order moments can be decoupled considering that the profile function $p(x,t)$ can be written as

$$p(x,t) = p_0(x,t) + \delta p(x,t)\,. \tag{4}$$

Based on simulational evidence, the quantity δp is necessarily small and fast. Consequently, we keep terms to lowest order in δp and we get

$$\frac{\partial p_0}{\partial t} = \frac{\partial^2 p_0}{\partial x^2} + p_0(1-p_0)(a' + c' p_0)\,, \tag{5}$$

where $a' = a + \varepsilon(0)$ and $c' = 1 - 2\varepsilon(0)$.

We should remark here that this approach, despite the fact of being valid, in principle, for small noise, does not correspond to a systematic perturbative expansion in the noise intensity, as distinct effects of the noise are dealt with differently. Results coming from our lowest order approximation do contain contributions from all orders in the noise intensity, and therefore constitute a partial resummation of such an expansion.

We see in Eq. (3) that the mean front profile obeys a dynamic equation similar to the deterministic one Eq. (1) but with renormalized parameters. We can now calculate the selected shape and velocity of the front, within the present approximation, by using the known results for deterministic equations of this type.

The multiplicative noise increases the control parameter $a' > a$ and so the strength of the linear term, and it reduces the weight of the nonlinearities of the deterministic model through the parameter $c' < 1$. Hence one could expect an increase of the propagating velocity and an increase of the domain of validity of the linear-marginal-stability criterion as the intensity of the noise is increased.

As a simple linear stability analysis indicates, for $a > -\varepsilon(0)$, the $\psi = 0$ state is unstable and then a continuum of propagating velocities is possible. The minimum of them is given by the different linear and nonlinear-marginal-stability criteria as mentioned above. Specifically, the linear regime (L) is now delimited by the control parameter range $\frac{1}{2} - 2\varepsilon(0) \le a < 1$ (Fig. 2). In this range, any initial profile that asymptotically falls off more quickly than $e^{-\kappa_l x}$ with $\kappa_l = \sqrt{a + \varepsilon(0)}$, propagates with the long time asymptotic velocity

$$v_l = 2\sqrt{a + \varepsilon(0)} \tag{6}$$

and a decay κ_l. On the other hand, the nonlinear regime (NL) holds for $-\varepsilon(0) \le a < \frac{1}{2} - 2\varepsilon(0)$ (Fig. 2). Here the long time asymptotic propagation velocity for initial profiles with

$$k \ge \kappa^* = \frac{a + \varepsilon(0)}{\sqrt{\frac{1}{2} - \varepsilon(0)}}$$

is given by

$$v_{\rm nl} = \frac{2a + 1}{\sqrt{2[1 - 2\varepsilon(0)]}}\,, \tag{7}$$

which decays with a $\kappa_{\rm nl} = \sqrt{\frac{1}{2} - \varepsilon(0)}$.

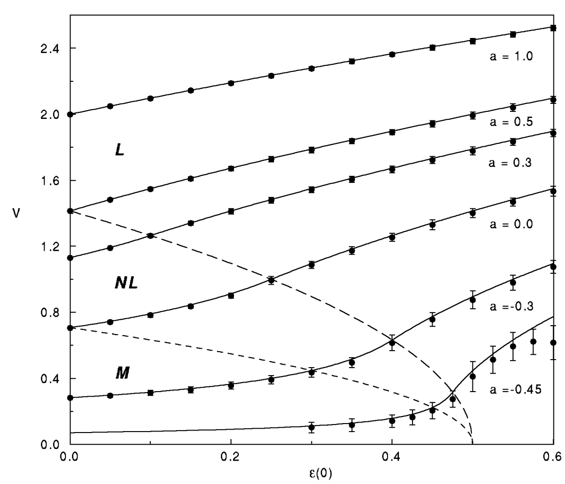

FIG. 2. Front velocities vs noise intensity $\varepsilon(0)$ for several values of a. Continuous lines display analytical predictions. Dashed lines divide the different regimes. Points and their error bars correspond to numerical simulation starting from steplike profiles ($\Delta x = 0.5$ and $\Delta t = 10^{-2}$).

The multiplicative noise reduces the metastable regime (M) to the range $-\frac{1}{2} \leq a < -\varepsilon(0)$ (Fig. 2). In this case, a unique front solution is allowed with a propagating velocity given by Eq. (7) and $\kappa_{\rm n1}$.

The minimum velocity of the noisy fronts, Eqs. (6) and (7), is shown in Fig. 2. The different regimes (L, NL, and M) are also displayed and it is shown how the linear criterion extends its range of validity as the intensity of the noise is increased. The agreement of the theoretical prediction with the simulation results is remarkable even for large values of the intensity of the noise.

Finally, we can establish that propagating velocities higher than the former minimum, Eq. (6) or (7), are also accessible. Initial fronts with $k < \kappa_l$ in the linear regime or with $k < \kappa^*$ in the nonlinear regime, propagate with a velocity $v = \frac{k^2+a'}{k}$. The asymptotic averaged front profile is e^{-kx} as $x \to \infty$; that is, these fronts keep their initial spatial asymptotic decay. The increase of the propagating velocity for these "quenched stochastic solutions" is greater as their initial k is smoother.

In order to test this prediction, numerical simulations have been performed with $a = 0.1$ and several values of the noise intensity $\varepsilon(0)$, starting from an initial slow decaying profile with $k = 0.1$. The numerical propagation velocities are compared in Fig. 3 with the analytical predictions, showing a very good agreement.

In summary, we conclude that, for the class of equations studied, the standard scenario of front propagation still holds in the presence of external multiplicative noise. A very good approximation even for moderately large noise intensities is given by an effective equation for the mean front of the same form as the original without noise, but with modified parameters. The main result is then that noise enlarges the domain of validity of the linear marginal stability zone. An interesting consequence of this result is that multiplicative noise may sustain quenched solutions within a continuous range of propagation velocities when the system would be, in the absence of noise, in the metastable regime, $-\frac{1}{2} \leq a < 0$, where a unique profile propagating with a single velocity $v_{\rm n1}$ would be allowed. Noise can thus stabilize profiles and velocities which are not allowed by the deterministic problem [16].

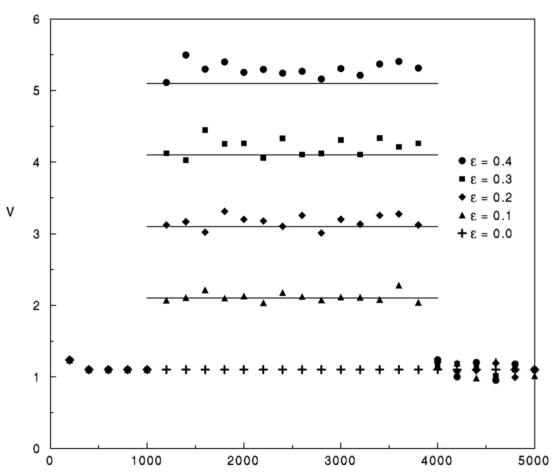

FIG. 3. Velocities for fronts with initial slow decaying profile $k = 0.1$ and different noise intensities. Lines display analytical predictions and symbols correspond to numerical simulations. At $t = 1000$, noise is applied and at $t = 4000$ the noise is switched off.

We acknowledge financial support by Comisión Interministerial de Ciencia y Tecnología (Projects PB93-0769, PB93-0759, PB93-0054) and Comissió Interdepartamental de Recerca i Innovació Tecnològica (Project GRQ-1993-01004), and computing support by Fundació Catalana per a la Recerca through Centre de Supercomputació de Catalunya.

[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993), and references therein.
[2] G. Dee and J.S. Langer, Phys. Rev. Lett. **50**, 1583 (1983). See also J.S. Langer, in *Chance and Matter*, edited by J. Souletie, J. Vannimenus, and R. Stora (North-Holland, Amsterdam, 1987).
[3] E. Ben-Jacob, H. Brand, G. Dee, L. Kramer, and J.S. Langer, Physica (Amsterdam) **14D**, 348 (1985).
[4] W. van Saarloos, Phys. Rev. Lett. **58**, 2571 (1987); Phys. Rev. A **37**, 211 (1988).
[5] W. van Saarloos, Phys. Rev. A **39**, 6367 (1989).
[6] J.A. Powell, A.C. Newell, and C.K.R.T. Jones, Phys. Rev. A **44**, 3636 (1991).
[7] G.C. Paquette, L.Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. Lett. **72**, 76 (1994); L.Y. Chen, N. Goldenfeld, and Y. Oono, Phys. Rev. E **49**, 4502 (1994).
[8] W. van Saarloos, M. van Hecke, and R. Holyst, Phys. Rev. E **52**, 1773 (1995).
[9] G.F. Mazenko, O.T. Valls, and P. Ruggiero, Phys. Rev. B **40**, 384 (1989).
[10] H.P. Breuer, W. Huber, and F. Petruccione, Physica (Amsterdam) **73D**, 259 (1994).
[11] A. Lemarchand, A. Lesne, and M. Mareschal, Phys. Rev. E **51**, 4457 (1995.)
[12] J. Riordan, C.R. Doering, and D. ben-Avraham, Phys. Rev. Lett. **75**, 565 (1995).
[13] L. Schimansky-Geier and Ch. Zülicke, Z. Phys. B **82**, 157 (1991).
[14] F. de Pasquale, J. Gorecki, and J. Popielawski, J. Phys. A **25**, 433 (1992).
[15] P.D. Ronney, B.D. Haslam, and N.O. Rhys, Phys. Rev. Lett. **74**, 3804 (1995).
[16] J. Armero *et al.* (to be published).
[17] L. Ramirez-Piscina, J.M. Sancho, and A. Hernández-Machado, Phys. Rev. B **48**, 119 (1993); **48**, 125 (1993).
[18] Our simulations show that the effective diffusion coefficient for the spreading of the front position increases with noise intensity but the actual dependence may not be trivial. This point is currently under study and will be discussed elsewhere.
[19] E.A. Novikov, Zh. Eksp. Teor. Fiz. **47**, 1919 (1964) [Sov. Phys. JETP **20**, 1290 (1965)].