Numerical methods for dissipative systems
Part 1: ODEs

1. Using the method of undetermined coefficients for the local truncation error, show that the most accurate explicit $s = 2$-LMSF is

$$v^{n+1} + 4v^n - 5v^{n-1} = k(4f^n + 2f^{n-1}),$$

with order of accuracy $p = 3$.

2. Consider the scalar initial-value problem

$$u_t(t) = \mathrm{e}^{\cos(t\,u(t))}, \quad t \in [0,3], \quad u(0) = 0.$$

   a) Approximate the solution by means of three different methods, *Euler explicit*, 4th-order *Adams-Bashforth* and 4th-order *Runge-Kutta*. For each method, reduce the time step k until you are confident that your solution is accurate to at least 6 digits. Use the RK4 method to compute the starting values of the AB4 method.

   b) For each method, use the *'exact'* reference value $u(3)$ obtained in a) with 6 digits of accuracy to plot the error $\varepsilon = |v(3) - u(3)|$ on a *log-log* scale as a function of k as well as a function of the *number of evaluations* of $f(u,t)$. Which method is the most efficient?.

3. Write a code to plot stability region boundaries for AB, AM and BDF methods of arbitrary order. Define a variable $\nabla = 1 - z^{-1}$ that takes values over a N-dimensional grid of points $z_n = \mathrm{e}^{\mathrm{i}2\pi n/N}$ over the unit circle ($N = 200$ or so). The code must compute each LMSF independently, so do not use coefficient tables for α_j or β_j.

   a) Plot the stability regions of AM for $p = 1, 2, 3$. Describe what happens for $p = 4$.

   b) Same as a) for BDF and $p = 1, 2, 3, 4$. Describe what happens for $p = 7$.

   c) Plot the stability region of the optimal formula shown in Problem 1. Explain.

4. A very popular integrator is the usually termed as *Crank-Nicolson*, given by the recurrence

$$v^{n+1} = v^n + \frac{k}{2}(f^{n+1} + f^n).$$

   a) What is the stability region of the method? Is it A-stable?

   b) In the linear approximation ($f = \lambda u$), show that

$$v^n = \left(\frac{1 + \bar{k}/2}{1 - \bar{k}/2}\right)^n v^0,$$

   with $\bar{k} = \lambda k$. Explain the disadvantages of this formula in the stiff situation where $\bar{k} \to -\infty$.

   c) Consider the initial value problem

$$u_t = -200(u - \cos t), \quad u(0) = 0, \quad t \in [0,1].$$

   Aproximate the solution by means of the Crank-Nicolson integrator and also with the *Backwards Implicit Euler* method ($v^{n+1} = v^n + kf^{n+1}$). Take, say, $k = 3 \cdot 10^{-2}$ and explain the initial oscillations of the CN approximation in terms of what you concluded in b). Apply the same argument to explain the absence of such oscillations with the Euler method.

For more problems and theoretical details, see: *Finite Difference and Spectral Methods for Ordinary and Partial Differential Equations*, by L. N. Trefethen (unpublished). Downloadable from *http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/pdetext.html*