

Consider a system of nine particles of unit mass and negligible size which are *fixed* on a two dimensional grid located at the points  $\mathbf{r}_1 = (x_1, y_1) = (0, 0)$ ,  $\mathbf{r}_2 = (x_2, y_2) = (1, 0)$ ,  $\dots$ ,  $\mathbf{r}_9 = (x_9, y_9) = (2, 2)$  as represented in the figure. We will assume that each mass creates a Newtonian two-dimensional gravitational field so that the total force acting on another particle of unitary mass located at a point  $\mathbf{r} = (x, y)$  of the plane is:

$$\mathbf{F}(x, y) = - \sum_{j=1}^9 \frac{\mathbf{r} - \mathbf{r}_j}{\|\mathbf{r} - \mathbf{r}_j\|^3}.$$

(A) Compute the line integral  $I = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$  along the path  $\Gamma_1$  (see figure) parametrized as follows:

$$\Gamma_1 : \mathbf{r}(\gamma) = \left( \frac{3}{2} + \frac{3}{4} \cos(\pi\gamma), \frac{3}{2} + \sin(2\pi\gamma) \right), \quad \gamma \in [0, 1].$$

For the approximation, make use of a 2nd order composite Newton-Cotes formula and a Gauss-Legendre quadrature formula. Check the accuracy of your results. In a log-log scale plot, represent the quadrature error as a function of the number of points of your discretization.

**Hint:** for your convergence analysis remember that  $\mathbf{F} = -\nabla\Phi$ , where  $\Phi(x, y)$  is a potential field.

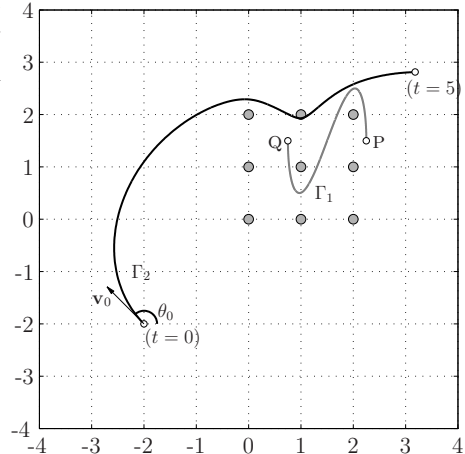
Consider another particle of unitary mass initially located at the point  $\mathbf{r}_0 = (-2, -2)$  moving with initial velocity  $\mathbf{v}_0 = (-2^{-1/2}, 2^{-1/2})$ . The dynamics of the particle is given by Newton's 2nd law:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} = (F_x, F_y),$$

that can be written as a system of first order differential equations:

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{u} = F_x, \quad \dot{v} = F_y,$$

with initial conditions  $x(0) = -2$ ,  $y(0) = -2$ ,  $u(0) = -2^{-1/2}$  y  $v(0) = 2^{-1/2}$ . The figure shows the resulting trajectory  $\Gamma_2$  of the particle for  $t \in [0, 5]$ .



(B) Compute the coordinates  $(x, y)$  and velocity  $(u, v)$  of the particle at  $t = 5$  with, at least, five exact digits. Plot the trajectory in the plane as the one shown in the figure. Use an accurate integrator such as a 4th order Runge-Kutta scheme.

(C) Starting from the same initial coordinate as in previous section, find two trajectories passing through the point  $(3, 3)$ . In this case, consider two initial shoots with different initial speeds. Consider a first shoot with  $v_0 = \|(u_0, v_0)\| = 0,7$  and a second one with  $v_0 = \|(u_0, v_0)\| = 1,5$ . For both shoots you have to find the initial angle  $\theta_0$  in radians and the instant of time at which the particle passes through the point  $(3, 3)$ . Provide  $\theta_0$  and  $T$  with 4 digits of accuracy for both shoots.

**Hint:** in this part we recommend you to find the zeros of the function

$$\mathbf{f}(\theta_0, T) = (x(T) - 3, y(T) - 3),$$

that depends on the total integration time  $T$  of the shoot and its initial angle  $\theta_0$ . You will have to use a Newton method with suitable initial guesses that you will have to find by inspection.