Effect of finite width on the onset of binary fluid convection

A. Alonso*, **O. Batiste**

Universitat Politècnica de Catalunya
Departament de Física Aplicada, Campus Nord, mòdul B5, 08034 Barcelona, Spain
e-mail: arantxa@fa.upc.es

ABSTRACT

The purpose of this work is to investigate the influence of the side walls on the onset of convection in a horizontal rectangular cavity filled with a binary mixture when heated from below. For the first time the three dimensionality of the problem is taken into account without making any approximation and considering realistic boundary conditions. In previous numerical works the width of the cell was either considered to be infinity (bulk mixtures) or different approximations usually valid in the narrow cell limit were assumed (i.e. Hele-Shaw and non-ideal Hele-Shaw approximations). The results we find show that the presence of the walls has a considerable effect on the onset of convection even for intermediate transverse aspect ratio cells. Surprisingly, they also show that the approximations generally assumed fail to reproduce the correct behaviour of the critical parameters even for quite narrow cells. We have compared the critical values of the Rayleigh number and the frequency that we obtain with those reported in the literature [1] and we find a quantitatively agreement within the experimental error.

INTRODUCTION

Convection in binary fluids mixtures has become a paradigmatic experimental system for the study of nonlinear wave phenomena and pattern formation [2]. What makes this system interesting is the fact that the first instability can be oscillatory, so the complex phenomena that arise can be described using weakly nonlinear theories based on perturbations of the basic state. Nevertheless, things are in fact more complicated due to the subcritical character of the bifurcation, and the validity of the perturbation approach is then not so obvious.

Most of the experiments have been made using narrow convection cells in order to suppress three dimensional instabilities. Some of them use long rectangular cells with aspect ratios typically ranging from 20 to 40 [3],[4]; others use annular containers with large radius ratios (≈ 60) [1],[5]. The aspect ratios in the transverse dimension are usually in the range 1 to 6 with some of them being even narrower (≈ 0.3) [6],[7].

The evolution equations that describe this system are well known and validated, and have been used for the derivation of amplitude equation models and in full numerical simulations that compare very well with experiments [8]. But in all cases the transverse dimension has been suppressed or its effect has been introduced using, for instance, the non-ideal Hele-Shaw approximation [9]. At present there are no quantitative predictions of the effect of the walls available even for the linear case.

In this work we present a linear stability analysis of the basic state for a binary fluid mixture contained in a narrow three dimensional rectangular cell. We will solve the full three dimensional equations with periodic boundary conditions in the long dimension and experimental boundary conditions in the other walls. The use of periodic boundary conditions corresponds to an annular experimental cell (the curvature has no effect here due to the large radius ratio used in experiments), but the values of the critical parameters are also correct for sufficiently large rectangular cells.

MATHEMATICAL FORMULATION OF THE PROBLEM

We consider a box of infinite horizontal length (x-direction) with height d (z-direction) and width b (y-direction) filled with a Boussinesq binary fluid of thermal and mass diffusivities κ and D, kinematic viscosity ν, thermal and concentration expansion coefficients α and β, and Soret coefficient S_T. The box is heated from below in the presence of vertical gravity $\mathbf{g} = -g\hat{\mathbf{e}}_z$. There exists a basic stationary conduction state with vertical temperature and concentration gradients,

$$\mathbf{u}_c = 0, \tag{1a}$$

$$T_c = T_0 - \Delta T(\frac{z}{d} - \frac{1}{2}), \tag{1b}$$

$$C_c = C_0 + C_0(1 - C_0)S_T\Delta T(\frac{z}{d} - \frac{1}{2}), \tag{1c}$$

with ΔT being the imposed temperature difference and T_0 and C_0 the mean temperature and concentration. The stability of the conduction state is described by the Navier-Stokes, continuity and heat equations which, once nondimensionalized by using the height of the layer d as lengthscale, d^2/κ as timescale and ΔT as temperature scale, take the form

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \sigma\nabla^2\mathbf{u} + [Ra(1+S)\Theta + \sigma RaS\eta]\hat{\mathbf{e}}_z, \tag{2a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2b}$$

$$\partial_t\Theta + (\mathbf{u}\cdot\nabla)\Theta = w + \nabla^2 T, \tag{2c}$$

$$\partial_t\eta + (\mathbf{u}\cdot\nabla)\eta = -\nabla^2\Theta + \tau\nabla^2\eta. \tag{2d}$$

Here, $\mathbf{u} = (u, v, w)$ is the velocity field, Θ denotes the departure of the temperature from its conduction profile, $T = \Theta + T_c$, and $\eta = C - C_0 - \Theta$. The Rayleigh, Prandtl, and Lewis numbers and the separation ratio are defined by

$$Ra = \frac{\alpha\Delta T g d^3}{\kappa\nu}, \qquad \sigma = \frac{\nu}{\kappa}, \qquad \tau = \frac{D}{\kappa}, \qquad S = C_0(1-C_0)\frac{\beta}{\alpha}S_T.$$

We have considered no-slip boundary conditions for the velocity in all the walls, perfectly conducting top and bottom walls and insulating front and back walls

$$\mathbf{u} = \mathbf{n}\cdot\nabla\eta = 0 \qquad \text{in } \partial\Omega \tag{3a}$$

$$\Theta = 0 \qquad \text{in } z = 0, 1 \tag{3b}$$

$$\partial_y\Theta = 0 \qquad \text{in } y = 0, \Gamma_y \tag{3c}$$

where $\Gamma_y = b/d$ is the transverse aspect ratio. The equations are solved numerically with a pseudospectral method using a formulation based on velocity potentials [10] in which the velocity field is written as

$$\mathbf{u} = \nabla\times(\xi\hat{\mathbf{e}}_y + \Psi\hat{\mathbf{e}}_z).$$

The spatial dependence of the variables has been expanded in terms of basis $A_l(y)$ and $B_m(z)$ for the y and z dependence, which are combinations of the Chebyshev polynomials that satisfy the boundary conditions, and Fourier expansions for the periodic direction

$$\hat{\Psi}(x, y, z) = e^{ikx}\sum_{l,m}\alpha_{klm}A_l(y)B_m(z).$$

To determine the critical values of the Rayleigh number, the corresponding frequency and the wavenumber of the pattern, we look for solutions of the linearized equations of the form

$$\Psi(x,y,z,t) = \hat{\Psi}(x,y,z)e^{(s+i\omega)t},$$

and then we minimize the stability curve of the Rayleigh number as a function of the wavenumber k that we obtain imposing the condition $s=0$ for any choice of the parameter values.

RESULTS

We have chosen as reference values for our computations the parameters of the fluid and the geometry of the box presented in [1]. The cell they consider is a long, narrow annulus with a radius ratio of $\Gamma_y = 1.288$ in width and $\Gamma_x = 67.09$ in mean circumference. The binary fluid is a water-ethanol mixture with $\sigma = 9.16$, $\tau = 0.008$ and $S = -0.257$.

In figure 1 we explore the dependence of the critical Rayleigh number on the transverse aspect ratio of the container and we compare the results we obtain by solving the exact three dimensional stability problem (thick solid line) with those resulting from the non-ideal Hele-Shaw approximation (thin solid line) and from the Hele-Shaw limit (dashed line). In the Hele-Shaw approximation the term $\nabla^2\mathbf{u}$ in the Navier-Stokes equation is replaced by $-12\mathbf{u}/w^2$, thus reducing the problem to two-dimensions. In the non-ideal Hele-Shaw approximation the effect of finite width is taken into account by replacing the Laplacian in the viscosity term by $\nabla^2 = \partial_x^2 + \partial_z^2 - 12/w^2$. As expected, the critical values resulting from the Hele-Shaw approximation are only correct for extremely narrow cells. When the non-ideal Hele-Shaw approximation is considered, both the value of the critical Rayleigh number in the Hele-Shaw limit and that obtained in a large aspect ratio container (bulk mixture) are recovered, but remarkably the results we find show that the differences with respect to the values obtained with a real 3-D computation are significant for a wide range of intermediate aspect ratios. The marginal stability curves for the critical frequency and the selected wavenumber reveal even more important discrepancies in the behaviour predicted by the approximations and that found in the real 3-D computations.

The comparison with the experimental values presented in [1] (a square in the plot) shows an excellent agreement between the 3-D calculations and the experiment. Experimentally, the reported value of the

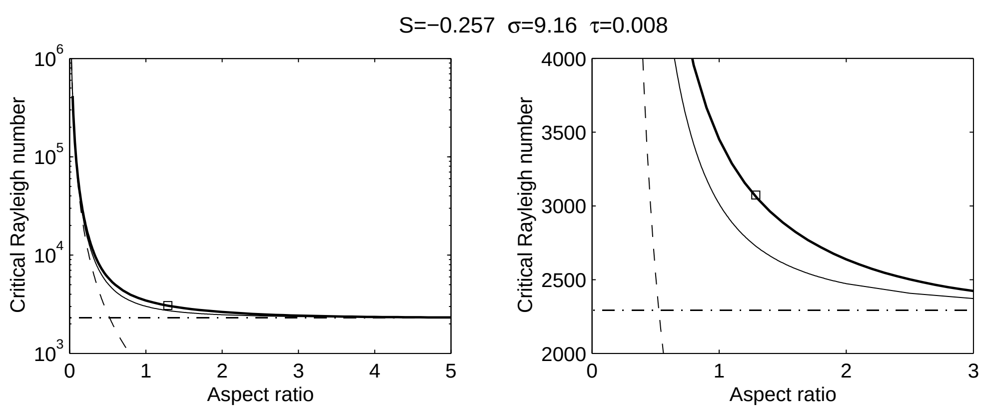

Figure 1: Neutral stability curves obtained by solving the 3-D stability problem (thick solid line) and by using the non-ideal Hele-Shaw (thin solid line) and the Hele-Shaw (dashed line) approximations. The dashed-dotted line shows the critical Rayleigh number for the bulk mixture and the square the one found in the experiment reported in [1].

critical Rayleigh number is $Ra_c^{\rm exp} = 3074$, with an error $\Delta Ra_c^{\rm exp} = 17$. In a bulk mixture theory predicts onset of convection at $Ra_c^{\rm b} = 2292$, which is quite far from the experimental values. The predictions are slightly improved when the non-ideal Hele-Shaw approximation is considered, as it gives $Ra_c^{\rm niHS} = 2726$, but the Rayleigh number is still $\approx 13\%$ smaller than observed. Finally, the 3-D stability analysis of the conduction state sets the onset of convection at a critical Rayleigh number $Ra_c^{\rm 3D} = 3063$, which falls within the experimental error. The 3-D computations also improve drastically the predicted values of the Hopf frequency. While the experimental critical frequency of the pattern is found to be $\omega_c^{\rm exp} = 13.8$, the approximations that might be suitable here give smaller and very similar values for this parameter. The value obtained in the non-ideal Hele-Shaw case is $\omega_c^{\rm niHS} = 11.5$ and that of a bulk mixture $\omega_c^{\rm b} = 11.4$. When the 3-D problem is solved the predicted value raises to $\omega_c^{\rm 3D} = 13.7$. So the agreement with the experimental results is excellent, although further comparison with other experiments would be of interest and will try to be made.

The analysis of the critical eigenfunctions also show that the presence of the lateral walls clearly affect the travelling waves that arise at the onset of binary fluid convection in an unbounded rectangular cell. The walls distort the convection rolls, which are assumed to be independent of the y-coordinate in 2-D computations, and the usually neglected y-velocity component seems to be significant even in intermediate transverse aspect ratio containers.

So our results make us think that there are some interesting features of the onset of binary convection in cells with transverse aspect ratios of order $\Gamma < 3$ that are being skipped in the non-ideal Hele-Shaw approximation, and obviously in the 2-D problem, that our computations can account for.

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