# On a subcritical transition scenario for pipe flow

**Álvaro Meseguer***

Departament de Fisica Aplicada
Universitat Politècnica de Catalunya, Campus Nord UPC, 08034 Barcelona, Spain
e-mail: alvar@fa.upc.es - Web page: http://www-fa.upc.es

**ABSTRACT**

In this work, a scenario of subcritical transition in Hagen-Poiseuille flow or pipe flow is presented. The analysis is focused on the *streak breakdown* process by which two-dimensional streamwise-independent finite amplitude perturbations transiently modulate the basic flow leading to a profile that contains saddle points and is linearly unstable with respect to very small streamwise-dependent perturbations. This mechanism is one possible route of transition to turbulence in subcritical shear flows. The exploration is carried out for initial disturbances of different finite amplitudes and axial and azimuthal periodicity. This study covers a wide range of Reynolds numbers and the double threshold curve obtained for transition is consistent with experimental observations.

## 1 Introduction

Hydrodynamic instability of pipe flow remains one of the oldest and yet unsolved problems of fundamental fluid dynamics. Pipe or plane Couette problems belong to a particular family of shear flows which are usually termed *subcritical* [2]. From a mathematical point of view, these flows are linearly stable, i.e., the spectrum of the linearized Navier-Stokes operator around the basic flow always lies on the stable half of the complex plane. Therefore, any infinitesimal perturbation added to the basic flow must eventually decay. Nevertheless, these flows become turbulent in the laboratory. For instance, below a critical Reynolds number, $\mathrm{Re}_c$, in the range $1760 \lesssim \mathrm{Re}_c \lesssim 2300$, pipe Poiseuille flow does not exhibit a sustained transition to turbulence [1]. For Reynolds numbers higher than $\mathrm{Re}_c$, a finite amplitude disturbance is required to destabilize the flow. Experimental and numerical evidence suggest that transition in pipe flow is extremely sensitive to the size and structure of the perturbations.

In a recent work [3], Zikanov analyzed the stability of the Hagen-Poiseuille flow by means of adding streamwise-independent finite amplitude perturbations to the basic flow and exploring the linear stability of the resulting time-dependent streaks with respect to infinitesimal streamwise-dependent disturbances. Zikanov concluded that this time-dependent flow was linearly unstable with respect to certain streamwise-dependent perturbations with a preferred axial periodicity, depending on the Reynolds number and the initial amplitude of the two-dimensional perturbation. This suggests that the streaks would eventually be distabilized leading to the usually termed *streak breakdown* scenario. Therefore, the main goal of this work will be to study the nonlinear time evolution of a particular type of perturbations in pipe flow and to identify the streak breakdown mechanism as a possible route to turbulence in this particular problem. We will focus our attention on the early stages of transition to turbulence; the study of fully developed turbulent flow is beyond our scope.

## 2 Mathematical formulation and numerical methods

We consider the motion of an incompressible viscous fluid of kinematic viscosity $\nu$ and density $\rho$. The fluid is driven through a circular pipe of radius $a$ and infinite length by a uniform pressure gradient, $\Pi_0$, parallel to the axis of the pipe. The motion of the fluid is governed by the incompressible Navier-Stokes equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\Pi_0}{\rho}\hat{\boldsymbol{z}} - \nabla p + \nu\Delta\mathbf{v} \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

where $\mathbf{v}$ is the velocity vector field, satisfying the no-slip boundary condition at the wall.

$$\mathbf{v}_{\text{pipe wall}} = \mathbf{0}, \tag{3}$$

A basic steady solution of (1), (2) and (3) is the so-called *Hagen-Poiseuille flow*

$$\mathbf{v}_{\mathrm{B}} = (u_{\mathrm{B}}\ ,\ v_{\mathrm{B}}\ ,\ w_{\mathrm{B}}) = \left(0\ ,\ 0\ ,\ U_{\mathrm{CL}}\left[1-\left(\frac{r}{a}\right)^2\right]\right),\quad U_{\mathrm{CL}} = -\frac{\Pi_0 a^2}{4\rho\nu}, \tag{4}$$

where $U_{\mathrm{CL}}$ is speed of the flow at the center axis. Henceforth, all variables will be rendered dimensionless using $a$ and $U_{\mathrm{CL}}$ as space and velocity units, respectively. The axial coordinate $z$ is unbounded since the length of the pipe is infinite. In what follows, we assume that the flow is axially periodic with period $b$. In the dimensionless system, the flow is confined in the domain $(r,\theta,z)\in[0,1)\times[0,2\pi)\times[0,Q=b/a)$, the basic flow takes the form $\mathbf{v}_{\mathrm{B}} = (u_{\mathrm{B}}, v_{\mathrm{B}}, w_{\mathrm{B}}) = (0\ ,\ 0\ ,\ 1-r^2)$, and the parameter which governs the dynamics of the problem is the *Reynolds number* $\mathrm{Re} = \dfrac{aU_{\mathrm{CL}}}{\nu}$. For the stability analysis, we suppose that the basic flow is perturbed by a solenoidal velocity field vanishing at the pipe wall

$$\mathbf{v}(r,\theta,z,t) = \mathbf{v}_{\mathrm{B}}(r) + \mathbf{u}(r,\theta,z,t),\quad \nabla\cdot\mathbf{u} = 0,\quad \mathbf{u}(r=1) = \mathbf{0}, \tag{5}$$

and a perturbation pressure field

$$p(r,\theta,z,t) = p_{\mathrm{B}}(z) + q(r,\theta,z,t). \tag{6}$$

On introducing the perturbed fields in the Navier-Stokes equations, we obtain a nonlinear initial-boundary problem for the perturbations $\mathbf{u}$ and $q$:

$$\partial_t\mathbf{u} = -\nabla q + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} - (\mathbf{v}_{\mathrm{B}}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v}_{\mathrm{B}} - (\mathbf{u}\cdot\nabla)\mathbf{u}, \tag{7}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{8}$$

Equations (7) and (8) are discretized via a spectral solenoidal Petrov-Galerkin scheme. For the time integration we have used a fourth-order implicit Backwards Differentiation method for the linear terms in combination with a fourth order explicit Adams-Bashforth method for the nonlinear ones. The time marching process was started with a fully explicit fourth order Runge-Kutta algorithm.

## 3 Main results

In this section we study the nonlinear evolution of very small 3D-perturbations superimposed to the 2D initial disturbances. In what follows, we define the *energy* of an arbitrary vector field $\mathbf{u}$ as the inner product

$$\mathrm{E}(\mathbf{u}) = \frac{1}{2}\int_0^Q \mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\mathrm{d}r\ \mathbf{u}^*\cdot\mathbf{u}. \tag{9}$$

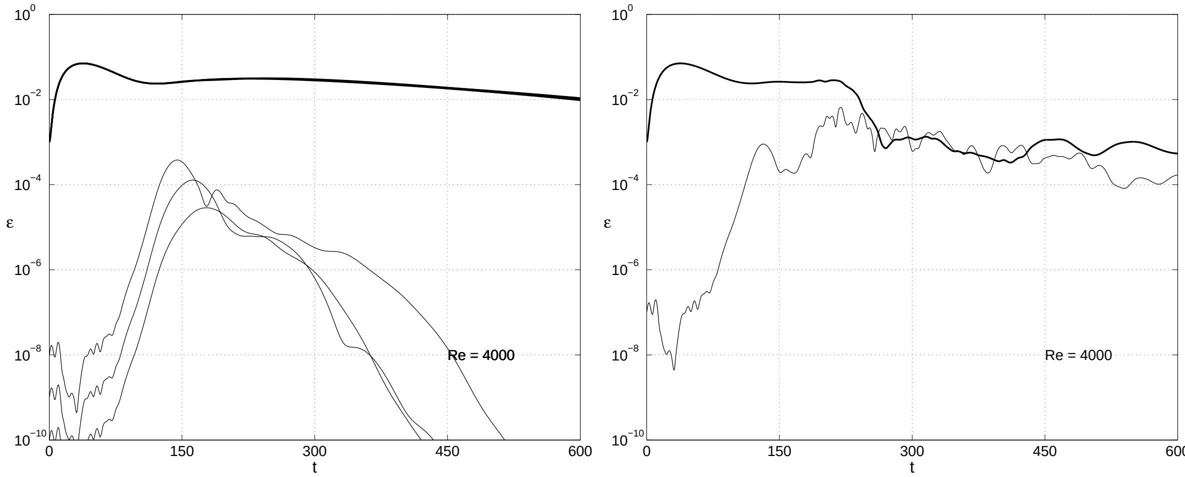


Figure 1: Formation of *streaks* (left) and *streak breakdown* mechanism (right).

We split the initial perturbation in two parts

$$\mathbf{u}^0_{\mathrm{S}} = \mathbf{u}^0_{2\mathrm{D}} + \mathbf{u}^0_{3\mathrm{D}}, \tag{10}$$

where $\mathbf{u}^0_{2\mathrm{D}}$ is the streamwise-independent component and $\mathbf{u}^0_{3\mathrm{D}}$ is the streamwise-dependent contribution, so that the energy of the 2D and 3D modes is distributed as follows:

$$\varepsilon(\mathbf{u}^0_{2\mathrm{D}}) = \varepsilon^{2\mathrm{D}}_0, \quad \varepsilon(\mathbf{u}^0_{3\mathrm{D}}) = \varepsilon^{3\mathrm{D}}_0, (\varepsilon^{3\mathrm{D}}_0 << \varepsilon^{2\mathrm{D}}_0). \tag{11}$$

In figure 1, we have carried out a computation for Reynolds number, $\mathrm{Re} = 4000$, and amplitude of the initial perturbation $\varepsilon^{2\mathrm{D}}_0 = 1 \cdot 10^{-3}$ (thick line). In the first case (figure 1-left), we see that three independent 3D-perturbations of energies $\varepsilon^{3\mathrm{D}}_0 = 1 \cdot 10^{-10,\,-9,\,-8}$ (thin lines), are not strong enough to destabilize the streak. We observe remarkable similarities of the evolution of the 3D-perturbations until they reach their maximum amplification, where the linear mechanisms are much stronger than the nonlinear ones. In this case, the threshold three-dimensional energy for this instability mechanism is of order $10^{-7}$, see figure 1, right.

In order to make a consistent comparison with the experimental time especifications, we considered time integrations in the interval $0 \le t \le T = 250$, i.e., the time required by a fluid particle located at the pipe axis to be advected downstream by the basic Hagen-Poiseuille flow a distance of 125 pipe diameters. According to the experiments of Darbyshire and Mullin, henceforth referred as D&M, the perturbation was injected 70 pipe diameters downstream of the pipe inlet and 120 upstream from the outlet. Even the mechanisms of transition presented here may slightly differ from the ones triggered in the experiments, each one of our numerical runs should cover the transition dynamics observed in the laboratory. We considered the same perturbations as the ones used in the previous sections, always starting with $\varepsilon^{3\mathrm{D}}_0 = 1 \cdot 10^{-7}$ and $\varepsilon^{2\mathrm{D}}_0$ ranging from $9 \cdot 10^{-6}$ to $2 \cdot 10^{-2}$. We included only one streamwise-dependent mode in the integration of axial wavenumber $k = 2\pi/Q = 1.5$ which is a good candidate to trigger transition according to Zikanov's linear computations. Our criteria of identification of sustained chaotic evolution was based on the energy of the 3D-perturbation at the end of each run. In particular, we classified the run as successful if $\varepsilon^{3\mathrm{D}}(t = T) > 1 \cdot 10^{-4}$, i.e., at the end of the run, the energy associated with the 3D-perturbation was still three orders of magnitude bigger than its initial value. No qualitative differences were found when increasing the number of spectral modes or reducing the time step. We carried out 740 runs whose final results have been outlined an compared with the experimental results in figure 2. To make a consistent comparison it is necessary

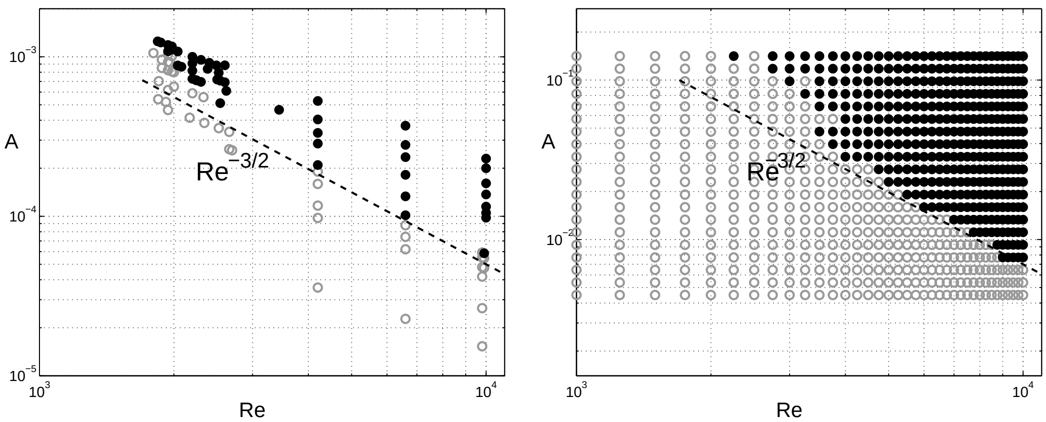


Figure 2: Experimental (left) and numerical (right) results.

to divide the D&M results by one power of $\mathrm{Re}$, giving $\gamma \sim -3/2$, and this has been done in figure 2(left). Thus figure 2(left) represents not the original plot from D&M [1] but a reprocessing of that data to make a comparison (the numbers on the vertical scale of figure 2(left) are subject to an arbitrary constant; thus the discrepancy of the vertical axes labels on the figure is not significant). In figure 2(left), black dots represent experimental transition to turbulence and white dots represent relaminarization of the flow within the pipe domain. In figure 2(right), we have represented the numerical results of our integrations. The amplitude $A = \sqrt{\boldsymbol{\varepsilon}_0^{\rm 3D} + \boldsymbol{\varepsilon}_0^{\rm 2D}}$ on the ordinate axis represents the square root of the total initial energy of the perturbation. White dots represent those situations where the flow relaminarized by the time the run was ended. Black dots represent successful transition. Despite the coarse numerical approximation of the problem, we observe a significant agreement with the experimental observations. First, according to our computations (figure 2-right), there is no transition for $\mathrm{Re} \lesssim 2000$. Second, the threshold amplitude decreases in both cases quite similarly. We have included a straight line in both plots representing the asymptotic curve $\mathrm{Re}^{-3/2}$ in order to compare the experimental and numerical behaviour. As we see, the agreement is very good. Nevertheless, it is probably too early to associate this apparent agreement with a common mechanism of transition. In both explorations, the Reynolds number is far from being within an asymptotic range. In fact, the analyses have been done still very close to the vertical threshold ($\mathrm{Re} \sim 2000$), thus strongly affecting the slope of the threshold boundary. To put it mildly, experiments and numerics are both providing the same *local* slope, none of them being asymptotic. In any case, the author *does not* claim that the streak breakdown is the only responsible for that transition by itself. Certainly, the mechanism explored here provides a consistent explanation if we compare with the experiments, but the real dynamics occurring in the laboratory are far from being understood.

## REFERENCES


[1] A. G. Darbyshire & T. Mullin, *Transition to turbulence in constant-mass-flux pipe flow.* J. Fluid Mech. **289**, 83-114, 1995.

[2] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability.* Cambridge University Press, Cambridge, 1981.

[3] O. Y. Zikanov, *On the instability of pipe Poiseuille flow.* Phys. Fluids **8**(11), 2923-29, 1996.