Instabilities of the Stewartson Layer

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ABSTRACT

In the classical Taylor-Couette problem of the flow between differentially rotating cylinders or spheres, the differential rotation rate $(\Omega_i - \Omega_o)$ is typically greater than the average rotation rate $(\Omega_i + \Omega_o)/2$, with $\Omega_o = 0$ (and hence $\Delta\Omega = 2\Omega_{av}$) being the simplest, textbook example.

In this work we will consider instead the limit where the overall rotation is very large, and the differential rotation is relatively small. Aside from being an interesting variant on the classical Taylor-Couette problem, this limit is also of considerable interest in geophysical fluid dynamics (e.g., oceanography or meteorology), in which a very rapid overall rotation is typically a dominant feature.

So, suppose we have a spherical shell in rapid overall rotation Ω, with additionally a differential rotation $\Delta\Omega$ imposed on the inner sphere. Scaling length by the gap width $(r_o - r_i)$, time by the inverse rotation rate Ω^{-1}, and $\mathbf{U}$ by $\Delta\Omega\,(r_o - r_i)$, the Navier-Stokes equation in the rotating frame becomes

$$\frac{\partial \mathbf{U}}{\partial t} + Ro\,\mathbf{U} \cdot \nabla\mathbf{U} + 2\,\hat{\mathbf{e}}_z \times \mathbf{U} = -\nabla p + E\nabla^2\mathbf{U}, \tag{1}$$

where the Ekman and Rossby numbers

$$E = \frac{\nu}{\Omega(r_o - r_i)^2} \qquad \text{and} \qquad Ro = \frac{\Delta\Omega}{\Omega}$$

measure the overall and differential rotation rates, respectively.

In this work we will consider numerically computed solutions of (1), in the limit $E \ll 1$ and Ro up to $O(1)$. We begin by computing the axisymmetric basic states, which turn out to yield this so-called Stewartson layer. We then compute the non-axisymmetric instabilities of this layer, and find them to be very different for $\mathrm{sgn}(Ro) = \pm 1$, that is, for positive versus negative differential rotation. We compare and contrast this difference with previous experimental and analytical results, and conduct a number of numerical tests to elucidate its origin.

Figure 1 shows how the angular velocity ω varies between 0 at the outer boundary and 1 at the inner. $Ro = 0$, corresponding to an infinitesimal differential rotation, and $E = 10^{-3.5}$ to 10^{-5}, corresponding to an increasingly rapid overall rotation. As E decreases, we see very clearly the emergence of an increasingly thin shear layer on the so-called tangent cylinder $\mathcal{C}$, with $\omega \approx 1/2$ inside $\mathcal{C}$ but $\omega = 0$ outside.

In order to understand why the solutions should arrange themselves in this peculiar fashion, we recall the well-known Taylor-Proudman theorem, stating that the flow in a rapidly rotating system will tend to align itself with the axis of rotation. More formally, take the curl of (1) and use $\frac{\partial}{\partial t}$, Ro, $E \ll 1$ (in fact $\frac{\partial}{\partial t} = Ro = 0$ in Fig. 1) to obtain

$$\frac{\partial \mathbf{U}}{\partial z} \approx 0.$$

With this result, the solutions in Fig. 1 follow quite naturally: For fluid columns outside $\mathcal{C}$, $\omega = 0$ is the appropriate boundary condition at both the upper and lower boundaries, so $\omega = 0$ everywhere will
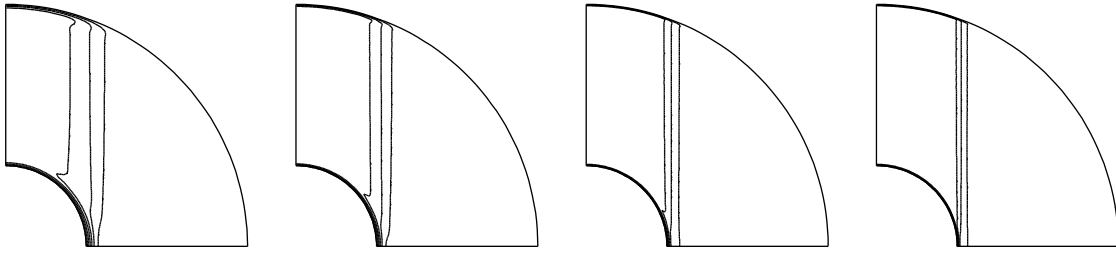
Figure 1: Contours of the angular velocity, for $Ro = 0$ and, from left to right, $E = 10^{-3.5}, 10^{-4}, 10^{-4.5}$ and 10^{-5}. The contour interval is 1/7.
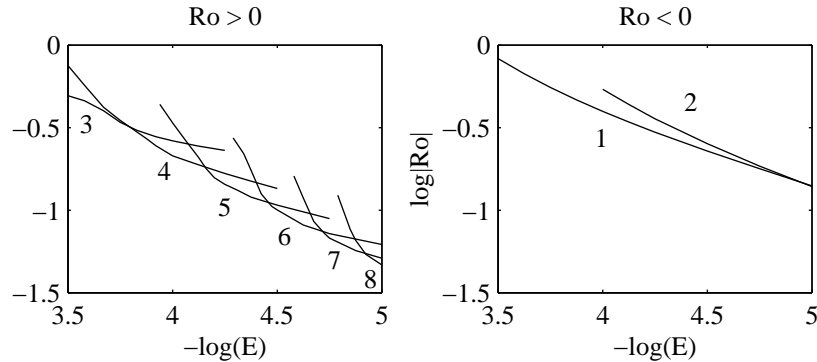

Figure 2: Ro_c versus E, for the different azimuthal wavenumbers indicated. Modes not shown may still be unstable, but are not the most unstable modes for any values of E.

satisfy both the Taylor-Proudman theorem as well as these boundary conditions. In contrast, for fluid columns inside $\mathcal{C}$, $\omega = 0$ is still the upper boundary condition, but the lower boundary condition is now $\omega = 1$. It is therefore not possible to satisfy the Taylor-Proudman theorem everywhere along the column. Instead, it is satisfied in the interior by having $\omega \approx 1/2$, with all the necessary z-dependence then concentrated into the so-called Ekman layers at the top and bottom boundaries.

Finally, the details of this shear layer that resolves this jump in ω across $\mathcal{C}$ were derived by Stewartson [1], who showed it to consist of a nested structure of innermost thickness $E^{1/3}$ right on $\mathcal{C}$, and outer layers $E^{2/7}$ just inside $\mathcal{C}$ and $E^{1/4}$ just outside. The results presented in Fig. 1 are broadly consistent with these scalings, even if the range of Ekman numbers shown there is not sufficient to clearly distinguish between these different sublayers. See, however, Hollerbach [2] or Dormy *et al.* [3] for detailed comparisons of numerical results with Stewartson's asymptotics.

Stewartson [1], Hollerbach [2] and Dormy *et al.* [3] all only considered the limit of infinitesimal differential rotation $Ro = 0$, in which case (1) is purely linear, and so the solution will necessarily remain axisymmetric. What happens for nonzero Ro though? These axisymmetric basic states in fact change remarkably little (that is, the inertial term $Ro\,\mathbf{U}\cdot\nabla\mathbf{U}$ is largely balanced by the pressure gradient $-\nabla p$). Nevertheless, as Ro is increased, the real, dimensional shear across the Stewartson layer increases correspondingly, so there must presumably come a point when it becomes unstable, by a mechanism similar to the classical Kelvin-Helmholtz instability. That is, we would expect it to roll up into a series of non-axisymmetric vortices.

Figure 2 shows these instability results, for positive and negative Ro. We note that the two cases are very different, with positive Ro yielding a progression to higher and higher azimuthal wavenumbers m as E is reduced, but negative Ro remaining at $m = 1$ over almost the entire range of Ekman numbers.
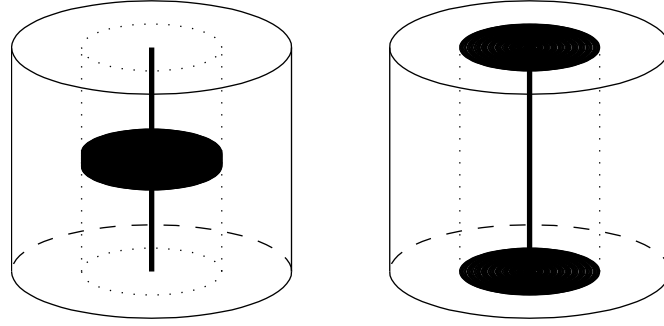
Figure 3: Sketches of the experimental setups of Hide & Titman on the left, and Früh & Read on the right. The disks indicated in black are given a differential rotation $\Delta\Omega$ in addition to the overall rotation Ω. The dotted lines denote the tangent cylinder on which the basic Stewartson layer forms.

This difference between $\text{sgn}(Ro) = \pm 1$ is particularly intriguing in light of the previous experimental studies (in cylindrical geometry) of Hide & Titman [4], who found this same difference, and Früh & Read [5], who did not, finding instead that positive and negative Ro both yield much the same results, namely this progression to higher and higher m. These latter results in turn are in agreement with Busse's [6] asymptotic analysis, which suggested there should be an exact symmetry between $\text{sgn}(Ro) = \pm 1$. We are therefore in the curious situation of having one experiment (Hide & Titman) and a numerical study (this one) which find there is a significant difference between positive and negative Ro, and having another experiment (Früh & Read) and an asymptotic study (Busse) which find there is not. The bulk of this presentation will therefore be devoted to elucidating the origin of this difference, and why it only manifests itself in some situations but not in others.

We begin by considering the experimental setups of Hide & Titman and Früh & Read, to see what differences there might be between the two. Figure 3 shows these setups, which we note are indeed different, with Hide & Titman having a single disk in the middle of a cylindrical tank, and Früh & Read having two disks embedded in the top and bottom boundaries of the tank. These geometrical differences then induce (subtle) differences in the axisymmetric basic states as well. We therefore consider the corresponding features in our spherical shell here (which we note is indeed more like 3a than 3b), and conduct a series of numerical experiments to determine which of them is the cause of this difference between $\text{sgn}(Ro) = \pm 1$ in some situations but not in others.

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