The Onset of Taylor-Like Vortex in the Flow Induced by an Impulsively Started Rotating Cylinder

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ABSTRACT

The onset of instability induced by impulsively started rotating cylinder was first investigated experimentally by Chen and Christensen [1]. The initial laminar flow evolves into a secondary flow pattern which consists of a series of Taylor-like vortices. In this transient boundary-layer system the critical time t_c to mark the onset of secondary motion becomes an important question. This problem may be called an extension of Taylor instability. The related instability analysis has been conducted by using amplification theory [2], the frozen-time model [2], and the maximum-Taylor-number criterion [3]. The first model requires the initial conditions and the criterion to define manifest convection. The second model is based on linear theory and yields the critical time as the parameter. These models take advantages of the similarity between Taylor instability and Rayleigh-Bénard instability.

Here we will extend propagation theory [4], which has been employed to analyze time-dependent Rayleigh-Bénard problem, into the instability of flow induced by an impulsively started rotating cylinder. The system considered here is a Newtonian fluid confined between the two concentric cylinders of radii R_i and $R_o(> R_i)$. Let the axis of inner cylinder be along the z' axis of a cylindrical coordinate system (r', θ, z'). At the time $t = 0$, the inner cylinder is impulsively started and maintained at a constant surface speed $V_0'(= R_i\Omega_i)$ and outer cylinder is kept stationary $\Omega_o = 0$. Here Ω_i and Ω_o are the angular velocities of inner and outer cylinder, respectively. The schematic diagram of the basic system is shown in Figure 1.

For a high V_0', secondary motion will set at a certain time before the flow becomes fully developed. The governing equations of the present flow field is expressed by

$$\nabla \cdot \mathbf{U} = 0, \tag{1}$$

$$\left\{\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right\}\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{U} \tag{2}$$

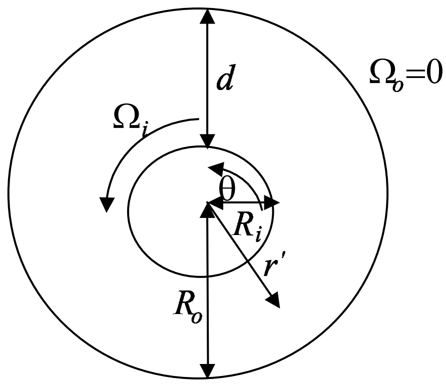


Figure 1: Schematic diagram of system considered here

where $\mathbf{U}$, P, ν and ρ represent the velocity vector, the dynamic pressure, the kinematic viscosity and the density respectively.

For small t, the basic velocity field is represented by

$$V_0 = \mathrm{erfc}\left\{\frac{y}{\sqrt{4\nu t}}\right\} \tag{3}$$

By neglecting the effect of curvature, *i.e.*, narrow-gap approximation, the above equations (1) and (2) can be linearized and the resulting dimensionless disturbance equations of tow-dimensional flow using equation (3) are represented by

$$\left(\frac{\partial^2}{\partial y^2} - a^2 - \frac{\partial}{\partial \tau}\right)\left(\frac{\partial^2}{\partial y^2} - a^2\right) u = 2V_0 a^2 v \tag{4}$$

$$\left(\frac{\partial^2}{\partial y^2} - a^2 - \frac{\partial}{\partial \tau}\right) v = \mathrm{Ta} u \frac{\partial V_0}{\partial y} \tag{5}$$

with proper boundary conditions,

$$u = \partial u/\partial y = v = 0 \qquad \text{at } y = 0 \text{ and } 1 \tag{6}$$

where $\tau = \nu t/d^2$, $u = d^2 u'/(\nu R_i)$, $v = v'/V_0'$, $V_0 = V/V_0'$, $y = (r - R_i)/d$ and $d = R_o - R_i$. The subscript '0' denotes the basic state and a represents the dimensionless vertical wavenumber. It should be noted that the radial velocity component u' is nondimensionalized by $\nu R_i/d^2$ rather than V_0'. In the present system the most important parameter is the Taylor number Ta defined as

$$\mathrm{Ta} = \frac{V_0'^2 d^3}{\nu^2 R_1} \tag{7}$$

Based on the balance between viscous and Coriolis terms, we set $u = \tau u^*(\zeta)$ and $v = v^*(\zeta)$. For a boundary-layer flow system of $\delta \propto \sqrt{\tau}$, the dimensionless time τ plays dual roles of time and length. Here δ denotes the boundary-layer thickness. Now, the self-similar stability equations are obtained from equations (4) and (5) as

$$\left\{\left(D^2 - a^{*2}\right)^2 + \frac{1}{2}\left(\zeta D^3 - a^{*2}\zeta D + 2a^{*2}\right)\right\} = 2V_0 a^{*2} v^* \tag{8}$$

$$\left(D^2 + \frac{1}{2}\zeta D - a^{*2}\right) = \mathrm{Ta}^* u^* V_0 \tag{9}$$

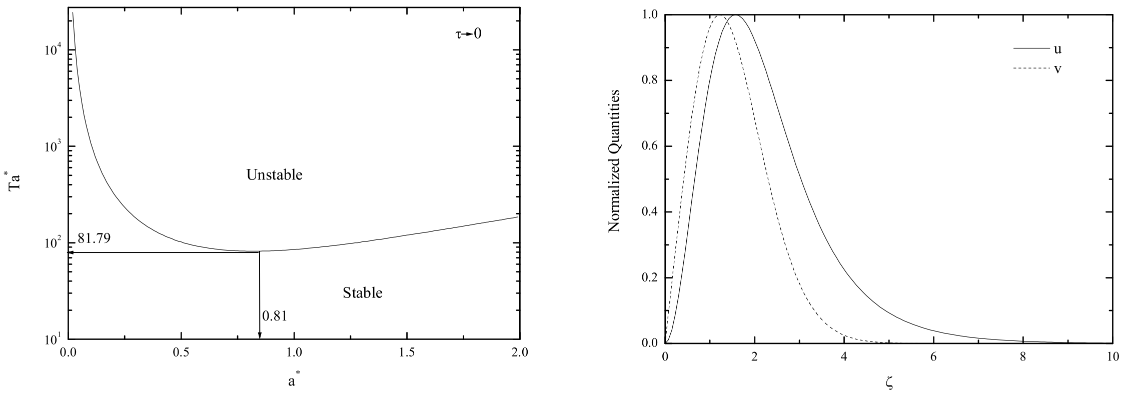
where $\zeta = y/\sqrt{\tau}$, $D = d/d\zeta$, $\mathrm{Ta}^* = \tau^{3/2}\mathrm{Ta}$ and $a^* = a\sqrt{\tau}$. The proper boundary conditions of no-slip are

$$u^* = Du^* = v^* = 0 \qquad \text{at } \zeta = 0 \text{ and } \infty \tag{10}$$

For a given τ, Ta^* and a^* are treated as eigenvalues and the minimum value of Ta^* should be found in the plot of Ta^* vs. a^* under the principle of exchange of stabilities.

By using outward shooting method with Newton-Raphson iteration, we solve the above stability equations and obtain the marginal stability curve. Based on the result of Figure 2(a), the critical conditions to mark the onset of secondary motion is given by

$$\tau_c = 18.84\mathrm{Ta}^{-2/3} \quad \text{and} \quad a_c = 0.19\mathrm{Ta}^{1/3} \qquad \text{for } \tau \to 0. \tag{11}$$



(a) (b)

Figure 2: Instability conditions for small time of $\tau_c \to 0$ from propagation theory; (a) marginal stability curve and (b) amplitude profiles at $\tau = \tau_c$

The resulting normalized amplitude functions of u^* and v^* are shown in Figure 2(b). For a given Ta, a fastest growing mode of infinitesimal disturbances would be set in at $\tau = \tau_c$ with $a = a_c$. The above equations show that τ_c decreases with an increase in Ta. Figure 3 illustrates that the present predictions of $4\tau_c$ $(\eta \to 1)$ compares well with Liu's [5] experimental data $(\eta = 0.2)$ marking the detection of manifest motion. Here η represents the ratio (R_i/R_o). The agreement of experimental data with the amplification theory and Tan and Thorpe's model [3] is also good but the latter model requires further justification.

Shen [6] suggested the momentary instability condition: the temporal growth rate of the perturbation quantity (r_1) should exceed that of the base flow (r_0). In the present system the dimensionless growth rates are defined as the root-mean-squared quantities of angular velocity components:

$$r_0 = \frac{1}{< V_0 >}\frac{d < V_0 >}{d\tau} \quad \text{and} \quad r_1 = \frac{1}{< v' >}\frac{d < v' >}{d\tau} \tag{12}$$

where $< quantity >= \sqrt{\left(\int_A (quantity)^2 dA\right)/A}$ and $A = Sdr'$ with $S = \pi d/a_c$. From the distributions of the base flow (equation (3)) and the perturbation quantities, we can obtaine the folowing relation:

$$r_0 = r_1 = \frac{1}{4\tau_c} \quad \text{for} \ \ \tau \to 0 \tag{13}$$

The above equation indicates that propagation theory bounds the momentary stability conception.

Foster [7] commented that $\tau_o \cong 4\tau_c$ for the time-dependent Rayleigh-Bénard problem. This means that a fastest growing mode of instabilities, which set in at $\tau = \tau_c$, will grow with time until manifest convection is detected at $\tau = \tau_o$. Chen and Kirchner [2] reported similar trend for the present time-dependent flow system. According to their results, the time of intrinsic instability $(\tau = \tau_i)$, *i.e.*, the time at which the disturbances first tend to grow, is about one-fourth of the time at which the instability motion is clearly observable experimentally. A growth period will be required, as illustrated in Figure 3. This scenario is supported by the results from the amplification theory (τ_i and $\tau_3 (= \tau_o)$). A more refined study including η-effect is now in progress.

It seems evident that during $t_c \le t \le t_o (\cong 4t_c)$ the cell size is almost constant but its vertical growth

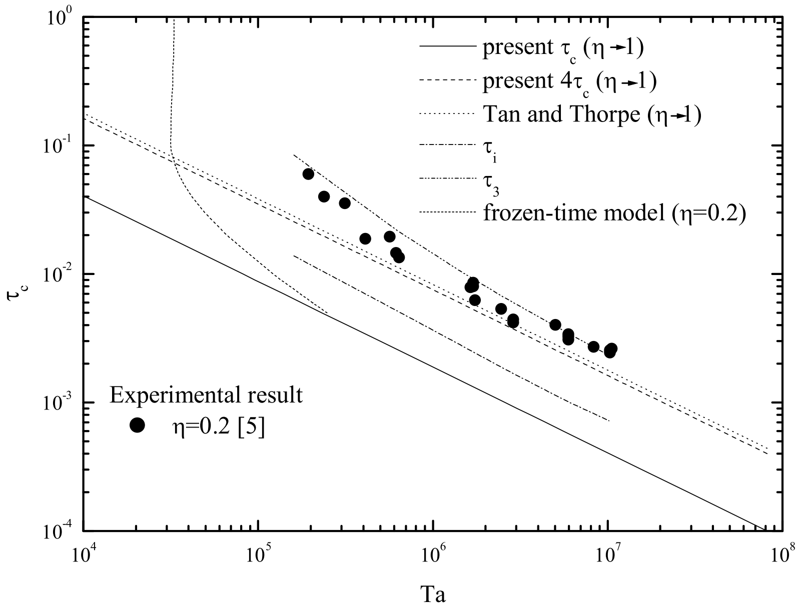


Figure 3: Comparison of predictions with experimental data of $\eta = 0.2$; For $\eta = 0.1$ τ_i and τ_3 from Chen and Kirchner [2].

will be continued. It is concluded that propagation theory yields the instability criteria compatible with experimental results in diffusive systems, hydrodynamical or thermal.

ACKNOWLEDGEMENTS

This work was supported by Cheju National University Development Foundation and LG Chemical, Seoul.

REFERENCES

[1] C.F. Chen, D.K. Christensen *Stability of Flow Induced by an Impulsively Started Rotating Cylinder*, Phys. Fluids **10**, 1845-1846, 1967.

[2] C.F. Chen, R.P. Kirchner *Stability of Time-Dependent Rotational Couette Flow. Part 2. Stability Analysis,* J. Fluid Mech. **48**, 365-384, 1971.

[3] K.-K. Tan, R.B. Thorpe *Transient Instability of Flow Induced by an Impulsively Started Rotating Cylinder,* Chem. Eng. Sci. **58**, 149-156, 2003.

[4] M.C. Kim, H.K. Park, C.K. Choi *Stability of an Initially, Stably Stratified Fluid Subjected to a Step Change in Temperature,* Theoret. Comput. Fluid Dynamics **16**, 49-57, 2002.

[5] D.C. Liu *Physical and Numerical Experiments on Time-Dependent Rotational Couette Flow,* Ph.D. Thesis, Rutgers University, New Jersey, 1971.

[6] S.F. Shen *Some Considerations on the Laminar Stability of Time-Dependent Basic Flows,* J. Aero. Sci **28**, 397-417, 1961.

[7] T.D. Foster *Onset of Manifest Convection in a Layer of Fluid with a Time-Dependent Surface Temperature,* Phys. Fluids **12**, 2482-2487, 1969.