Superfluid Couette flow in an enclosed annulus

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ABSTRACT

As the temperature is reduced through $T = T_\lambda = 2.1768\text{K}$, liquid helium undergoes a phase transition from helium I (a classical Navier-Stokes fluid) to helium II (a superfluid). At atmospheric pressure, this superfluid state persists through to absolute zero. In this region, helium II may be modelled using Landau's two-fluid theory, in which the fluid is considered to be made up of two completely mixed fluids; a viscous normal fluid and an inviscid superfluid. The total density of the fluid is given by $\rho = \rho_n + \rho_s$ where the normal and superfluid densities, ρ_n, ρ_s are temperature dependent. At absolute zero the normal fluid component is zero and helium II is entirely superfluid ($\rho_n = 0$), whilst at the transition point $T = T_\lambda$ the superfluid component is zero ($\rho_s = 0$) and the fluid behaves as a classical Navier-Stokes fluid. What is missing from Landau's theory is vortex lines, which appear in the superfluid if helium II is rotated with angular velocity Ω greater than some critical value. These vortex filaments are quantised, in that

$$\oint_C \boldsymbol{v_s} \cdot \boldsymbol{dl} = \Gamma \tag{1}$$

where $\boldsymbol{v_s}$ is the superfluid velocity field, $\Gamma = 9.97 \times 10^{-4}\text{cm}^2/\text{sec}$ is the quantum of circulation (the ratio of Plank's constant and the mass of one helium atom) and C is an arbitrary integration path around the axis of the filament. The vortices tend to align themselves to the direction of rotation and the number of vortices per unit area is found to be $N = 2\Omega/\Gamma$.

A macroscopic model for modelling helium II is known as the Hall-Vinen-Bekharevich-Khalatnikov (HVBK) equations, see [1, 2] for the derivation. These equations are derived assuming flow configurations in which there are a large number of vortex lines aligned in roughly the same direction. For such situations, the superfluid vorticity, which is discrete in nature, may be approximated as a continuum resulting in a superfluid vorticity field $\boldsymbol{\omega_s} = \text{curl}\, \boldsymbol{v_s}$. The isothermal, incompressible HVBK equations may be written as:

$$\frac{\partial \boldsymbol{v_n}}{\partial t} + (\boldsymbol{v_n} \cdot \boldsymbol{\nabla})\boldsymbol{v_n} = -\boldsymbol{\nabla} p_n + \nu_n \nabla^2 \boldsymbol{v_n} + \frac{\rho_s}{\rho}\boldsymbol{F}, \tag{2}$$

$$\frac{\partial \boldsymbol{v_s}}{\partial t} + (\boldsymbol{v_s} \cdot \boldsymbol{\nabla})\boldsymbol{v_s} = -\boldsymbol{\nabla} p_s + \boldsymbol{T} - \frac{\rho_n}{\rho}\boldsymbol{F}, \tag{3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v_n} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{v_s} = 0. \tag{4}$$

where $\boldsymbol{v}_n$ is the normal fluid velocity, p_n, p_s are effective pressures and ν_n is the kinematic viscosity of the normal fluid. The mutual friction force may be written as

$$\boldsymbol{F} = \tfrac{1}{2}B\widehat{\boldsymbol{\omega}}^s \times (\boldsymbol{\omega}^s \times (\boldsymbol{v}_n - \boldsymbol{v}_s - \nu_s \boldsymbol{\nabla} \times \widehat{\boldsymbol{\omega}}_s)) + \tfrac{1}{2}B'\boldsymbol{\omega}_s \times (\boldsymbol{v}_n - \boldsymbol{v}_s - \nu_s \boldsymbol{\nabla} \times \widehat{\boldsymbol{\omega}}_s) \tag{5}$$

with $\widehat{\boldsymbol{\omega}}_s = \boldsymbol{\omega_s}/|\boldsymbol{\omega_s}|$ the unit vector in the direction of superfluid vorticity and B, B' are the temperature-dependent mutual friction parameters [3]. This force is due to collisions between the normal fluid (rotons) and vortex lines. The vortex tension force may be written as

$$\boldsymbol{T} = -\nu_s \boldsymbol{\omega_s} \times (\boldsymbol{\nabla} \times \widehat{\boldsymbol{\omega}}_s) \tag{6}$$

and this models the energy in the vortex lines. The vortex tension parameter $\nu_s = (\Gamma/4\pi)\log(b_0/a_0)$ has the same dimension as kinematic viscosity but physically it is very different: it represents the ability of a vortex line to oscillate due to vortex waves which can be excited on the vortex lines themselves. The quantity $b_0 = (|\boldsymbol{\omega}_s|/\Gamma)^{-1/2}$ represents the intervortex spacing and a_0 is the radius of the vortex core.

In the case of flow between infinite cylinders Couette flow is an exact solution of the HVBK equations for both the normal fluid and superfluid. This is provided that Ω is greater than a small critical value at which vortex lines first appear. Barenghi & Jones [4] and Barenghi [5] performed a linear stability analysis on the HVBK equations. The Couette state was linearly perturbed and they numerically calculated the critical angular velocity, Ω_c and corresponding critical axial wavenumber at which Couette flow becomes linearly unstable. Their work prompted further experiments [6] and excellent agreement between the predicted and measured values were found particularly for temperatures close to the transition temperature [5]. At lower temperatures, although there was qualitative agreement, the experimentally measured critical Reynolds numbers were larger than the predicted values. This is thought to be due to the breakdown of the infinite cylinder assumption as a result of the elongation of the Taylor cells, as predicted by the stability analysis. The stretching of the Taylor cells is so pronounced that, in a typical apparatus, end effects will be important, even at relatively high temperatures.

In this work we extend previous results [7] in which we studied the Couette flow of helium II in a unit aspect ratio annulus. In particular we wish to investigate the Couette flow of helium II at varying aspect ratios and to see how it differs from the flow of a classical Navier-Stokes fluid. There are two more motivations to study the hydrodynamics of a superfluid. The first arises from the engineering applications: helium is the only substance available in liquid form at temperatures near absolute zero, so it is important as a cryogenics coolant. Applications range from infrared detectors in space science to the cooling of superconducting magnets in particle physics. The second motivation comes from recent experimental developments in which the relation between classical and quantum turbulence is investigated (for example see [8, 9]) including the construction of a superfluid wind tunnel.

We consider the fluid to be confined radially between two concentric cylinders of inner and outer radius R_1 and R_2, and axially between two fixed plates which are separated by a distance H. The top and bottom plates and the outer cylinder are held stationary and the inner cylinder rotates at constant angular velocity Ω. Throughout this work we shall consider the radius ratio, $\eta = R_1/R_2 = 0.976$ as in the experimental apparatus of Swanson & Donnelly [6] and will vary the Reynolds number, $\mathrm{Re} = \Omega R_1(R_2 - R_1)/\nu_n$, the aspect ratio, $h = H/(R_2 - R_1)$ and the temperature, T.

The boundary conditions for the normal fluid are no-slip, that is, working in cylindrical coordinates $\boldsymbol{v}_n = \Omega R_1 \widehat{\boldsymbol{\phi}}$ at $r = R_1$ and $\boldsymbol{v}_n = 0$ at $r = R_2$, $z = 0, z = H$. The boundary conditions for the superfluid are more delicate. Given $\boldsymbol{n}$, a normal to the boundary, we have that $\boldsymbol{v}_s \cdot \boldsymbol{n} = 0$ at $r = R_1, R_2$ and $z = 0, H$, ensuring no flow normal to the boundaries. For the remaining conditions we have taken

$$\boldsymbol{\omega}_s \times \widehat{\boldsymbol{z}} = 0 \quad \text{at} \quad r = R_1, R_2 \quad \text{and} \quad z = 0, H. \tag{7}$$

Thus the superfluid vorticity is purely axial at the boundaries. The condition on the cylinder walls, $r = R_1$ and $r = R_2$, has been discussed in a previous paper [10] whilst the condition on the ends of the cylinder, $z = 0$ and $z = H$, corresponds to perfect sliding of the vortex lines.

We make the simplifying assumption that the flow is axisymmetric and equations (2–4) are solved using a finite difference approach. The equations are discretised on a uniform grid in both the radial and axial direction and then stepped forward in time until a steady solution is achieved.

In previous work [7], in which we considered the Couette flow of helium II in a unit aspect ratio annulus, we found that the superfluid Ekman cells always rotate in a counter-classical manner. Whilst the normal fluid Ekman cells rotate in a classical direction at temperatures close to T_λ but reverse at lower temperatures. We extend these previous results to consider the flow of helium II at larger aspect ratios. In figure 1 we show how varying the temperature effects the flow of helium II at Re $= 100$ and $h = 4$. For all the plots the inner/outer cylinder is on the left/right respectively.

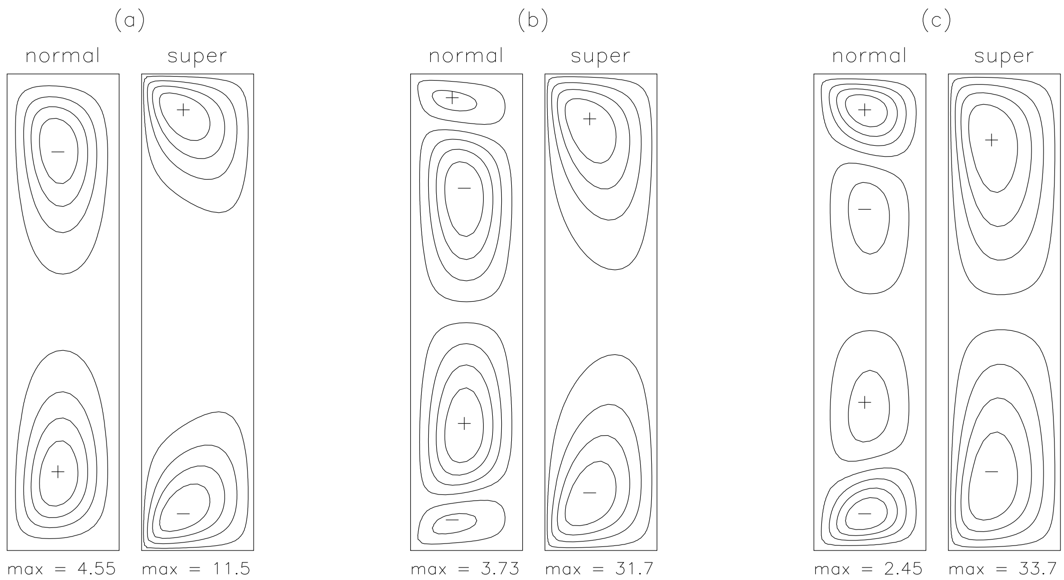

Figure 1: Plots of the stream function of the normal fluid and superfluid at Re $= 100$, $h = 4$ for (a) $T = 2.17$K, (b) $T = 2.14$K, (c) $T = 2$K. The maximum amplitude of the stream function is printed under each plot.

We can see that at $T = 2.17$K the situation is similar to that found at unit aspect ratio [7]. Both normal and superfluid have a pair of Ekman cells, with the normal fluid rotating classically, whilst the superfluid rotates counter-classically. As the temperature is reduced the superfluid continues to rotate counter classically whilst the normal fluid splits into four cells. This effect becomes more pronounced at lower temperatures. Thus the normal fluid always has outflow at the centre of the cylinders, whilst the superfluid has inflow there. Changing the temperature effects the relative flow close to the ends of the cylinders, which has been discussed previously [7].

In figure 2 we show how increasing the Reynolds number effects the flow of helium II at aspect ratio $h = 4$ and temperature $T = 2.16$K. We see that at Re $= 100$ both the normal and superfluid have a pair of Ekman cells, with the normal fluid rotating classically, and the superfluid rotating counter-classically. However as the Reynolds number is increased the superfluid splits into four cells, with the normal fluid staying as two cells. Thus it appears that the temperature determines the behaviour of the two fluids close to the ends of the cylinders, whilst as the Reynolds number increases the mutual friction forces the fluids to match at the centre.

Further examples of the anomalous motion of helium II in a rotating cavity will be discussed.

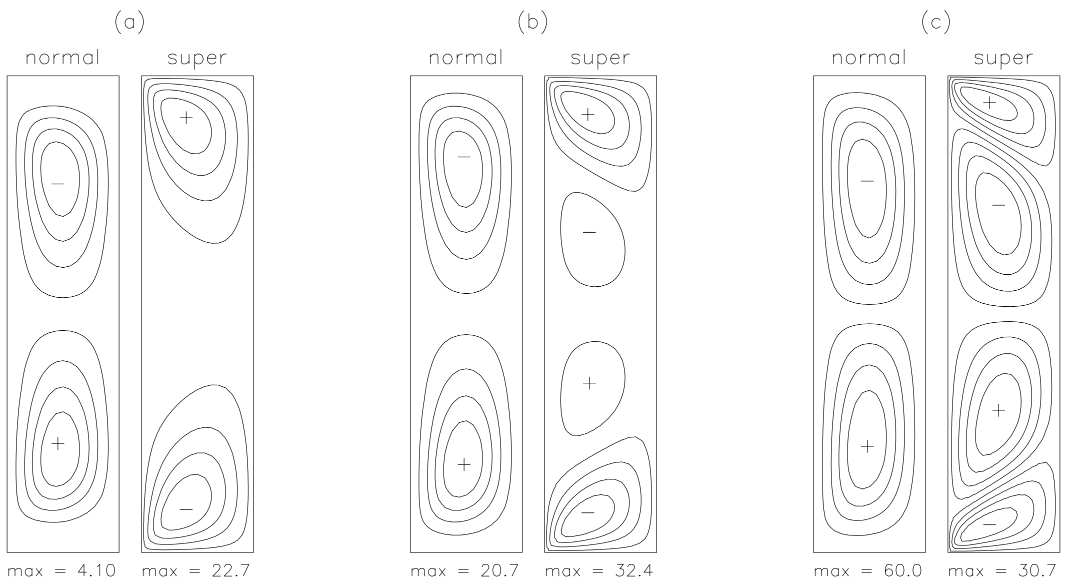

Figure 2: Plots of the stream function of the normal fluid and superfluid at $T = 2.16\text{K}$, $h = 4$ for (a) Re $= 100$, (b) Re $= 200$, (c) Re $= 300$. The maximum amplitude of the stream function is printed under each plot.

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