Unstable manifolds computation for the two-dimensional plane Poiseuille flow

Pablo S. Casas*, **Àngel Jorba**[1]

*Departamento de Matemática Aplicada I. Universidad Politécnica de Cataluña
Diagonal, 647; 08028 Barcelona. Spain
e-mail: pablo@casas.upc.es - Web page: http://www-ma1.upc.es/˜casas/

[1]Departamento de Matemática Aplicada y Análisis. Universidad de Barcelona
Gran Via, 585; 08007 Barcelona. Spain
e-mail: angel@maia.ub.es - Web page: http://www.maia.ub.es/˜angel/

ABSTRACT

In this work we study some aspects of the dynamics of the plane Poiseuille problem in dimension 2, in what refers to the connection among different configurations of the flow. The fluid is confined in a channel of plane parallel walls. The problem is modeled by the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\Delta\mathbf{u}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

being Re the Reynolds number, together with no-slip on the channel walls and L-periodic boundary conditions ($L = 2\pi/\alpha$, being α the parameter wave number). We have considered two different formulations to drive the fluid through the channel: holding constant the total flux or the mean pressure gradient. For each of them we obtain a different definition of $Re = hU_c/\nu$, where h represents half of the channel length, U_c the velocity of the laminar flow in the centre of the channel, and ν the kinematic viscosity. To be precise $Re_Q = 3Q/\nu$, $Re_p = Gh^3\rho/2\mu^2$, corresponds to the Reynolds numbers when we keep Q as a constant flux or G as a constant mean pressure gradient respectively. The fluid is supposed of constant density ρ and viscosity μ. The numerical approximation is detailed in [1]. Roughly, it employs Fourier and Chebyshev spectral discretizations of velocities and pressure in the periodic and transversal directions respectively. The temporal variable is approximated by means of finite differences.

In figure 1 we represent bifurcating curves of periodic and quasi-periodic flows obtained numerically in [1] using $N = 8$ and $M = 70$ spectral modes in the x and y spatial directions respectively. For the temporal discretizacion the time step has been prescribed to $\Delta t = 0.02$. Each point of those curves corresponds to the amplitude A (distance to the laminar flow in L_2-norm) of the flow (periodic or quasi-periodic) for the given value of Re. It is also marked on the curves the different stability regions, together with several Hopf bifurcations. In figure 1b at Re_{Q1}, the Hopf bifurcation of periodic flows give rise to a family of quasi-periodic solutions, whose stability is also presented. The main load of the computations has been carried out in parallel in a Beowulf cluster of PCs.

Unstable invariant manifolds of periodic flows

In our study of the dynamics of plane Poiseuille flow, we are going to analyse the connection between different configurations of the flow, such as laminar, periodic or quasi-periodic. We want to know how the fluid evolves and to which kind of solution it is conducted when it starts near an unstable periodic solution, as the ones shown in figure 1. We have selected several flows for $\alpha = 1.02056$ and $\alpha = 1.1$, taking for the spatial discretization $N = 8$ and $M = 70$ and for the temporal one $\Delta t = 0.02$. As [2] shows, periodic solutions of Poiseuille flow are stationary in a certain moving frame of reference, and therefore we study their stability by means of linearization. We consider unstable flows that have one real unstable eigenvalue and the remaining ones stable, or a couple of complex conjugate unstable eigenvalues and the remaining ones stable. The lower branch of periodic flows in figures 1a and 1b belong to the first group. In the case of constant pressure, the arc of the upper branch before Re_{p1} and

UPPER BRANCH									
$\alpha = 1.02056$		$\alpha = 1.02056$		$\alpha = 1.1$		$\alpha = 1.1$		$\alpha = 1.1$	
Re_p	attractor	Re_p	attractor	Re_p	attractor	Re_p	attractor	Re_Q	attractor
4638	laminar	8336	2-torus	3803	laminar	8839	2-torus	5264	2-torus
4654	laminar	8688	2-torus	3816	laminar	9316	2-torus	5402	2-torus
4680	laminar	9067	2-torus	3835	laminar	9832	2-torus	5601	2-torus
6952	2-torus	9478	2-torus	7268	2-torus	10388	2-torus	5801	2-torus
7184	2-torus	9921	2-torus	7615	2-torus	10990	2-torus	6069	2-torus
7438	2-torus	10398	2-torus	7991	2-torus	11638	2-torus	6321	2-torus
7713	2-torus	10912	2-torus	8398	2-torus			6589	2-torus
8012	2-torus							6682	2-torus
								6776	2-torus

LOWER BRANCH									
$\alpha = 1.02056$		$\alpha = 1.1$		$\alpha = 1.1$		$\alpha = 1.1$		$\alpha = 1.1$	
Re_p	attractor	Re_p	attractor	Re_p	attractor	Re_Q	attractor	Re_Q	attractor
4636	laminar	3802	laminar	7813	2-torus	3658	periodic	7359	2-torus
4649	laminar	3885	periodic	8192	2-torus	3694	periodic	7639	2-torus
4689	laminar	3969	periodic	8642	2-torus	3816	periodic	7995	3-torus
4722	laminar	4172	periodic	9094	2-torus	4020	periodic	8389^*	3-torus
4766	periodic	4570	periodic	9489	2-torus	4559	periodic	8682^*	unknown
4821	periodic	4872	periodic	9589	unknown	4611	periodic	9045	unknown
4890	periodic	5272	periodic	10513	2-torus	4814	periodic	9363	unknown
4975	periodic	5789	periodic	11078	laminar	5101	2-torus	9589	unknown
5079	periodic	6397	periodic	11375	2-torus	5500	2-torus	9848	unknown
5205	periodic	6917	periodic			5822	2-torus	10139	unknown
5361	periodic					6049	2-torus	10390	unknown
5554	periodic					6499	2-torus	10746	unknown
5772	periodic					6791	2-torus	11096	unknown
						7097	2-torus	11395	unknown

TABLE 1. Attractors of the flow to which is connected the unstable manifold of periodic solutions on the upper and lower branch of the amplitude curve. In all cases $N = 8$, $M = 70$ and $\Delta t = 0.02$. The attractors on the lower branch corresponds to one direction of the unstable manifold. On the opposite direction the attractor is the laminar solution, except for a few cases. See the text for details. The temporal evolution is presented in figures 2a and 2b for Re marked with '*' in the table.

after Re_{p2} for $Re_p < 14000$, correspond to the second group. On the other hand, for Re_Q, only the arc between Re_{Q1} and Re_{Q2} belongs to the second group. For each of those periodic flows $u^p(x, y, t)$, we have studied its unstable manifold and which new state of the fluid they are connected to. By means of the Jacobian matrix (the linearization of the discretized version of (1)) computed to analyze the stability, we can obtain the eigenvector $w \in \mathbb{C}^K$ (being K the dimension of the discretized system) associated with the unstable eigenvalue. The perturbed flow $u^p + rw$ for $|r| \ll 1$ is thus a first order approximation of the unstable manifold of u^p, which in turn is an attracting manifold. Using the numerical integrator of (1), we have followed the temporal evolution of $u^p + rw$ until an attracting state is reached. Likewise, in the case that u^p has only one real unstable eigenvalue, we have considered two ways of escaping from u^p namely, taking $r > 0$ or $r < 0$. When there is a couple of complex conjugate eigenvalues, we have chosen an arbitrary direction in the plane generated by the real and imaginary parts of w.

The different configurations obtained for several values of Re_p, Re_Q and α are summarized in table 1. The attractors presented in table 1 for the lower branch are obtained in one direction of the unstable manifold. On the opposite direction the attractor is the laminar solution, except for a few cases. For $Re_Q = 5822$, $\alpha = 1.1$, both directions of the unstable manifold are connected with a 2-torus. For $Re_p = 5772$, $\alpha = 1.02056$, both directions of the unstable manifold are connected with the periodic flow on the upper branch.

For the case of Re_p, $\alpha = 1.02056$, the lower branch is connected with the laminar flow for $Re_p \lesssim 4722$, and with a periodic solution for $4766 \lesssim Re_p \lesssim 5772$. For the case of Re_p, $\alpha = 1.1$, the connection of the lower branch is with the laminar flow for $Re_p \lesssim 3802$, with a periodic flow for $3885 \lesssim Re_p \lesssim Re_{p2}$, with a 2-torus for $Re_{p2} \lesssim Re_p \lesssim 9500$, and with different configurations for $Re_p \gtrsim 9500$. For Re_Q, $\alpha = 1.1$, the lower branch is connected with a periodic solution for $Re_Q \leqslant Re_{Q1}$, with a 2-torus

Re_Q	attractor	Re_Q	attractor	Re_Q	attractor
7953	3-torus	8322*	3-torus	8894	possible 3-torus
7975	3-torus	8486	3-torus	9005*	possible 3-torus
8043	3-torus	8623	3-torus	9096	unknown
8157	3-torus	8767	3-torus	9204	unknown
8278	3-torus				

TABLE 2. Attractors of the fbw to which is connected the unstable manifold of quasi-periodic solutions. In all cases $N = 8$, $M = 70$, $\alpha = 1.1$ and $\Delta t = 0.02$. The temporal evolution of the fbw until the attracting solution is reached is presented in fi gures 2c and 2d for Re marked with '*' in the table.

for $Re_{Q1} \leqslant Re_Q \lesssim 7950$, with a 3-torus for $7950 \lesssim Re_Q \lesssim 8500$, and with unknown sets for $Re_Q \gtrsim 8500$. In this case, for $Re_Q \geqslant Re_{Q1}$ the perturbed unstable periodic flow is first connected with the periodic solution on the upper branch and then is directed to the final attractor. The upper branch is connected with the laminar flow for $Re_p \leqslant Re_{p1}$, and with a 2-torus for $Re_{p2} \leqslant Re_p \leqslant Re_{p3}$, for $\alpha = 1.02056$ and, $\alpha = 1.1$, being Re_{p3} the next Hopf bifurcation after Re_{p2}. For the case of Re_Q, $\alpha = 1.1$, the upper branch is also connected with a 2-torus for $Re_{Q1} \leqslant Re_Q \leqslant Re_{Q2}$. In figures 2a and 2b we present the evolution of the perturbed periodic flow. On those figures we plot the projection of the discrete velocity on the plane of 2 coordinates (956 and 210 out of 1156, to be precise) when the flow crosses an appropriate Poincaré section (see [1] for details) Σ_1. For instance, on Σ_1 the evolution of a stable periodic flow is represented by a single constant point on those projections and the laminar flow by coordinates $(0, 0)$.

Unstable invariant manifolds of quasi-periodic flows

In the case of constant flux the bifurcation diagram of periodic flows is qualitatively different to that of constant pressure, as is shown in figures 1a and 1b. For Re_Q and $\alpha = 1.1$ there is a change of stability at the minimum Reynolds Re_{Q0} of the curve of amplitudes, but no new bifurcations are born there. The first Hopf bifurcation occurs at the point labeled Re_{Q1} in figure 1b.

The quasi-periodic solutions found from Re_{Q1} are stable for $Re_{Q1} < Re_Q \lesssim 7950$. At $Re_Q \approx 7950$ the branch of quasi-periodic solutions loses stability to give rise to a family of attracting tori of 3 basic frequencies at a new Hopf bifurcation.

In table 2 we show the connections of the unstable manifold corresponding to the unstable 2-tori for $Re_Q \gtrsim 7950$. The procedure follows the same lines as for the case of periodic flows described previously. For each unstable 2-torus u^q, we approximate the linear part of the Poincaré map P_c defined on Σ_1 (quasi-periodic flows are proved in [2] to be periodic in an appropriate moving frame and thus fixed points of P_c), by means of extrapolated finite differences. Then we perturb u^q in the direction of the most unstable eigenvectors of the linear part of P_c, associated to two complex conjugate eigenvalues of modulus greater than 1. We follow the temporal evolution of this perturbed flow until an attracting state is reached. For $Re_Q \lesssim 9000$ the attracting flow seems to be a 3-torus, but for greater Re_Q the solution becomes apparently disordered. In figures 2c and 2d we show the temporal evolution for the perturbed quasi-periodic flows for $Re_Q = 8322$ and $Re_Q = 9005$. The attracting solution for $Re_Q = 8322$ is a quasi-periodic flow with 3 basic frequencies close to resonant. In the case of $Re_Q = 9005$, the final attractor observed in the Poincaré section resembles a possible 3-torus.

REFERENCES

[1] Pablo S. Casas. *Numerical study of Hopf bifurcations in the two-dimensional plane Poiseuille flow*. PhD thesis, Universidad Politécnica de Cataluña, September 2002. http://www-ma1.upc.es/˜casas/research.html.

[2] D. Rand. Dynamics and symmetry predictions for modulated waves in rotating fluids. *Arch. Rat. Mech. Anal.*, 79:1–37, 1982.

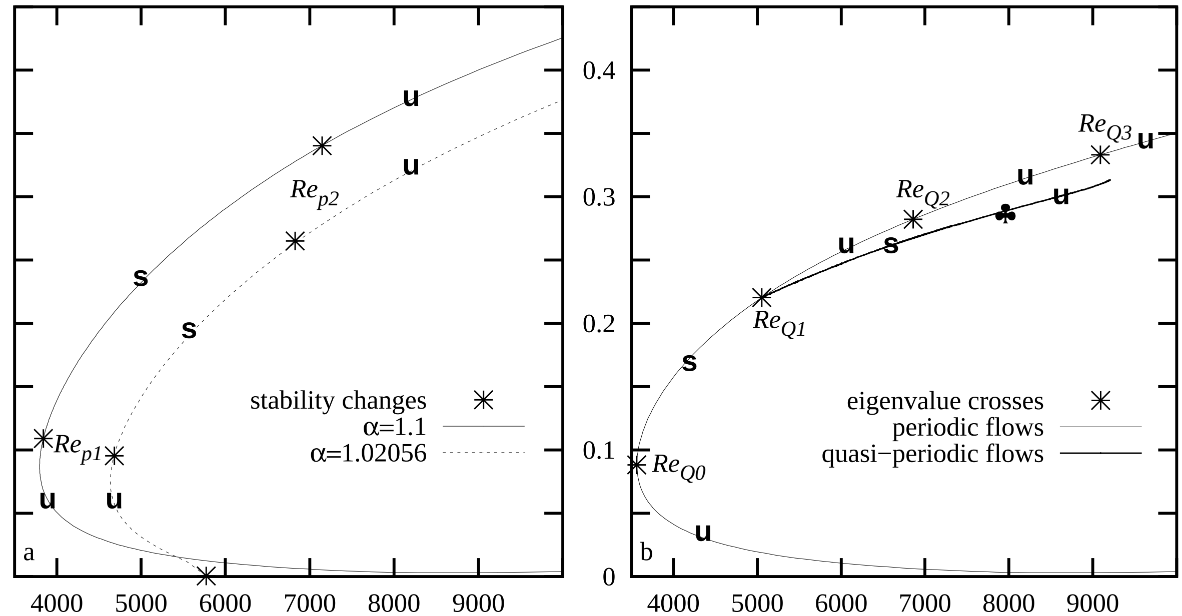

FIG. 1. Bifurcating curves of periodic flows based on Re_p (a) and Re_Q (b) in the horizontal axis and A in the vertical one. The number of retained spectral modes in the x and y direction are $N = 8$ and $M = 70$ respectively. The '*' on each curve of (a) corresponds to Hopf bifurcations; two of them are labeled as Re_{p1} and Re_{p2}. They divide the different regions of stability to superharmonic disturbances, which are also labeled in the plot as 's' for "stable" and 'u' for "unstable". The plot in (b) is the analogous of (a) based on Re_Q only for $\alpha = 1.1$ but including a branch of quasi-periodic flows. At Re_{Q0} there is no bifurcation whereas three Hopf bifurcations labeled as Re_{Q1}, Re_{Q2}, and Re_{Q3} are presented on the upper branch. The symbol ♣ indicates a Hopf bifurcation in the curve of quasi-periodic flows.

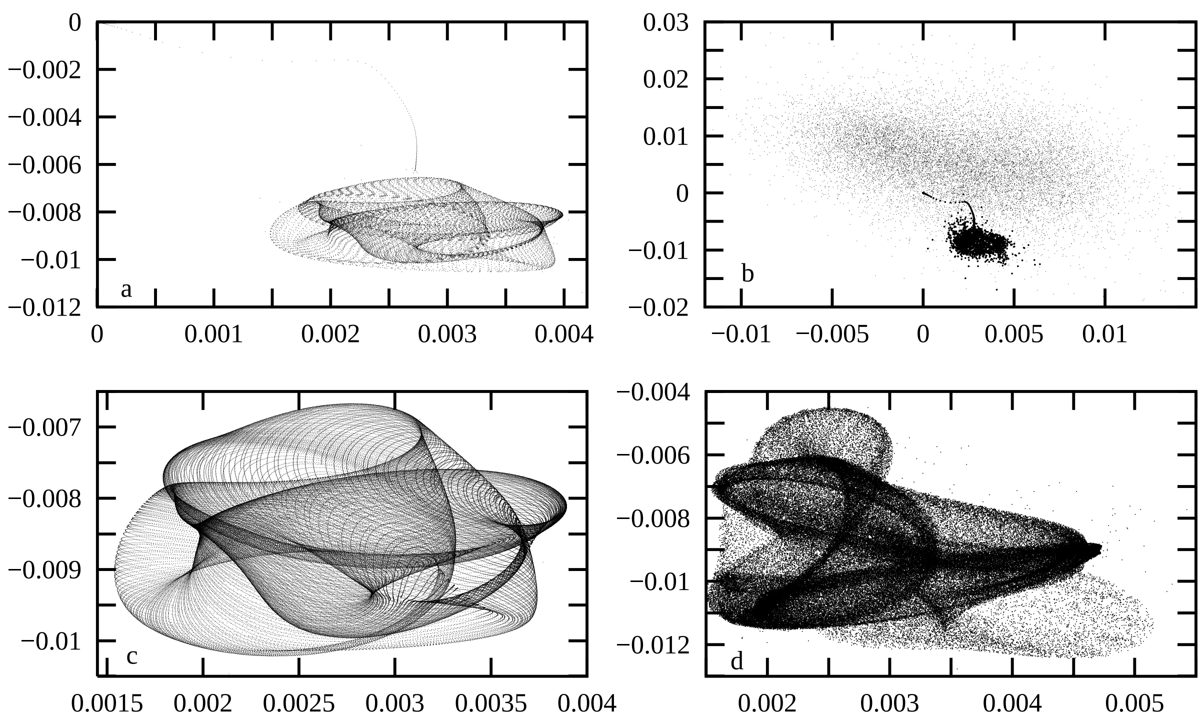

FIG. 2. Different time-evolutions of perturbed flows. In (a) and (b) the flow starts from the perturbed unstable periodic solution on the lower branch for $Re_Q = 8389$ and $Re_Q = 8682$ respectively, and $\alpha = 1.1$. In (a) the fluid is first directed to the unstable periodic solution on the upper branch and then attracted by a 3-torus. In (b) the fluid is attracted by a strange set plotted in bigger dots, which is unstable and finally drives the fluid to another strange set. In (c) the perturbed 2-torus for $Re_Q = 8322$ is attracted by a nearly resonant 3-torus, as can be observed. Finally in (d) the perturbed 2-torus for $Re_Q = 9005$ is attracted by a set that reminds a 3-torus.