Heteroclinic Cycles in the GEOFLOW-Experiment on the International Space Station (ISS)

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ABSTRACT

Introduction

The bifurcation theory with the $O(3)$ symmetry is a powerful tool in the spherical Bénard problem. But a such experiment cannot be realised on the earth beacause the earth gravity breaks the spherical symmetry. So, till now this bifurcation theory with the $O(3)$ symmetry has not been corroborated by the experiment. The GEOFLOW-experiment on the International Space Station is an opportunity to check this theory.

In this problem, there are two natural bifurcation parameters: one is the Rayleigh number R_a which is proportional to the buoyancy force responsible for the onset of convection; the second one is the aspect ratio η which is the ratio of the inner to the outer radius of the shell. When the Rayleigh number is increasing the "trivial" solution loses its stability. Generically, a unique spherical mode ℓ is unstable (codimension 1, only R_a is allowed to vary). In this case, we expect only stationary or travelling waves solutions [1]. For specific aspect ratio numbers η_c, two modes $(\ell, \ell+1)$ are unstable (codimension 2 bifurcation, R_a and η vary). For this mode interaction the dynamics are more complex. It is already known since the article of Guckenheimer and Holmes [2] that structurally stable heteroclinic cycles between group orbits of equilibria (i.e. steady states) can arise due to the symmetry of the problem. The (1,2) interaction was studied from a numerical point of view by Friedrich and Haken [3] and later by Armbruster and Chossat [4] using group theoretic methods. A general study of the $(\ell, \ell+1)$ interaction was presented in Chossat and Guyard [5]. The authors show, under certain "generic" conditions, that heteroclinic cycles of various types exist and these connections are "robust" against small perturbations. Furthermore, it has been proved [6] that the heteroclinic cycle does not completely destroy when the system is slowly rotating around an axis.

In this paper, we focus on the existence of heteroclinic cycles with the GEOFLOW-experiment requirements. We neglect the thin conductor wire in the shell and the domain does not rotate, so the problem has the $O(3)$ symmetry. The difference between the previous works is the pseudo-gravitational field: in our case it is a $1/r^5$ field instead of the classical Newtonian one. Thus, the existence requirements of heteroclinic cycles in [5] and [6] can be only numerically checked. In [7], we have found that for $\eta_c = 0.33$ and $\lambda_c = 19.8$ the $(2,3)$ interactions holds. Furthermore for $Pr = 0.24$, it occurs a degenaracy necessary to the existence of such heteroclinic cycles. In the following we study the codimension 2 bifurcation with these previous parameters values.

Mathematic Background

The gouverning equations for perturbation $\vec{v}$ of the velocity field of the fluid and Θ of the temperature field are set in the Boussinesq approximations. After a convinient choice of length scale, the spherical shell domain is $\Omega = \{\boldsymbol{r} \in \mathbb{R}, \eta < |\boldsymbol{r}| < 1\}$, and the dimensionless equations are given in this domain by:

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} &= -\nabla p + \Delta \overrightarrow{v} + \lambda \Theta g(r) \overrightarrow{r} - \vec{v}.\Delta \vec{v}, \\ \frac{\partial \Theta}{\partial t} &= \frac{1}{P_r}(\Delta \Theta + \lambda h(r) \overrightarrow{v}.\overrightarrow{r}) - \vec{v}.\nabla \Theta, \\ \nabla.\vec{v} &= 0. \end{aligned} \tag{1}$$

with the boundaries conditions: $\Theta(\eta) = \Theta(1) = 0$ (imposed temperature) and , $\vec{v}(\eta) = \vec{v}(1) = 0$ (viscous fluid). The functions $g(r) = \frac{1}{r^5}$ and $h(r) = \frac{1}{r^3}$ represent the dimensionless gravity field and the gradient of temperature respectively. P_r is the Prandtl number and λ is proportional to the square root of the Rayleigh number:

$$\lambda = \sqrt{\frac{2R_a}{(\frac{1}{\eta}-1)^3}} \tag{2}$$

The vector space V_ℓ of the ℓ^{th} critical mode has the dimension $(2\ell+1)$ and it is a irreductible representation space of $O(3)$. We shall denote $(\zeta_m^\ell)_{m=-\ell\ldots\ell}$ the complex eigenvectors of V_ℓ associated with the generalized spherical harmonics [7]. Then, for the interaction $(2,3)$, the eigenspace $V = V_2 \oplus V_3$ has the dimension 7+5=12 and a vector $U(t) \in V$ can be expressed as

$$U(t) = \sum_{m=-2}^{2} x_m(t)\zeta_m^2 + \sum_{n=-3}^{3} y_n(t)\zeta_n^3,$$

where $x_{-m} = (-1)^m \bar{x}_m$ and $y_{-n} = (-1)^n \bar{y}_n$.

The dynamics and bifurcation from the basic state near the critical values η_c and λ_c can be examined by varying the two system parameters λ and η. The original system of PDE's (1) is reduced on its center manifolds. This last one is parameterized by the space V. So, the bifurcation equations is governed by a system of ODE's which it consists of 5 equations for x_m and 7 equations for y_n. These equations admit, at any order, a Taylor expansion and an equivariant structure. In [7], the truncated equations are given at the third order and the coefficients are computed for a large range of the Prandtl number P_r.

For the Prandtl value $Pr = 0.24$, heteroclinic cycles can exist [7]. When only the mode 2 is unstable, we obtain two axisymmetric solutions $\xi^\pm$ (in $O(2) \oplus \mathbb{Z}_c^2$ isotropy subgroup) which differ with respect to the direction of the flow. These two kinds of flows are described by the amplitude x_0. On the line x_0, the solutions $\xi^\pm$ are stable for $\eta > \eta_c$ and $\lambda > \lambda_c$. Now, let us consider two isotropy subgroups Σ_1 and Σ_2 such their fixed point space contain x_0. For example, if ξ^- is a sink in $Fix(\Sigma_1)$ and a saddle in $Fix(\Sigma_2)$ and if ξ^+ is a saddle in $Fix(\Sigma_1)$ and a sink in $Fix(\Sigma_2)$, then it can exist branches which connect the both equilibria $\xi^\pm$. In our case, only the two different cycles are possible [5]:

- $(O(2) \oplus \mathbb{Z}_2^c, O(2)^-, D_6^d)$: type I.
- $(O(2) \oplus \mathbb{Z}_2^c, D_2 \oplus \mathbb{Z}_2^c, D_6^d)$: type II.

In order to determine the existence region of these cycles, we have to study the stability of $\xi^\pm$ in the fixed-point plan of the previous cycles. Furthermore, we have to check that for each plan the only equilibria are $\xi^\pm$ (no mixed solutions).

Results and Discussion

In agreement with [5], only the heteroclinic cycle of the type I has an open basin of attraction. Let us $\epsilon_\lambda = 100\frac{\lambda-\lambda_c}{\lambda_c}$ and $\epsilon_\eta = 100\frac{\eta-\eta_c}{\eta_c}$. The figure 1 shows in the plane of the parameters $(\epsilon_\lambda, \epsilon_\eta)$, the region of type I heteroclinic cycles. The both black lines delimit the stability of the trivial solution. The red line is the limit of the stability of ξ in regards y_0 ($Fix(O(2)^-)$), it is the most constraining requirement. The dotted black line is the limit of the existence of mixed solution in the different fixed-point spaces. An example of a heteroclinic cycle for $\epsilon_\lambda = -0.04$ and $\epsilon_\eta = -0.5$ is described. The figure 2 show that the transition time is short compared to the stay close to an equilibrium. Furthermore the dynamic stay more longer in the ξ^- than ξ^+. The evolution of the dynamic during the cycle is showed at the figure 3.

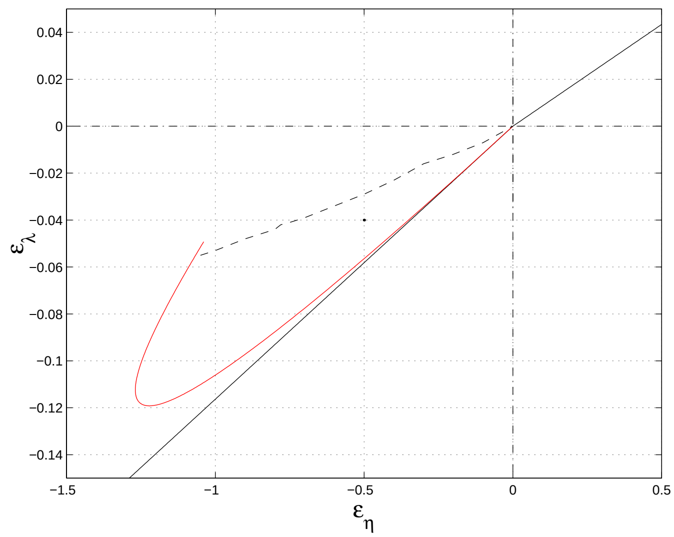

Figure 1: Heteroclinic cycles region delimited by the red and the black dotted lines.

At the relative time $t_r = 360$ the dynamic has the $O(2)^-$ symmetry and at $t_r = 872$ the dynamic has the hexagonal anti-symmetry: D_6^d.
In order to find a heteroclinic cycle, the relative variation of λ must be inferior to 0.12%. But, in the experiment we can obtain a such precision (1% for the temperature), so it will be very difficult to observe such phenemona. It would be interesting to determine the dynamics "far" from the bifurcation point.

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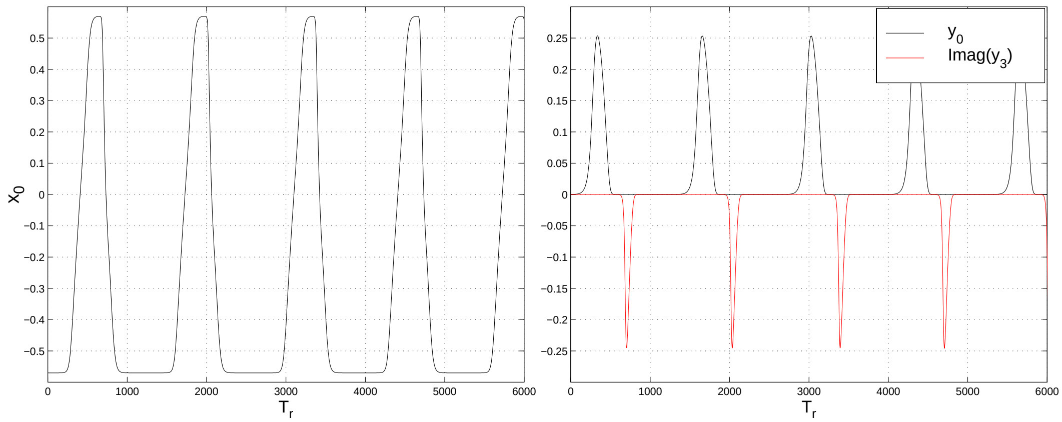

Figure 2: Time evolution of the amplitudes x_0 (a) and $Im(y_3), y_0$ (b)

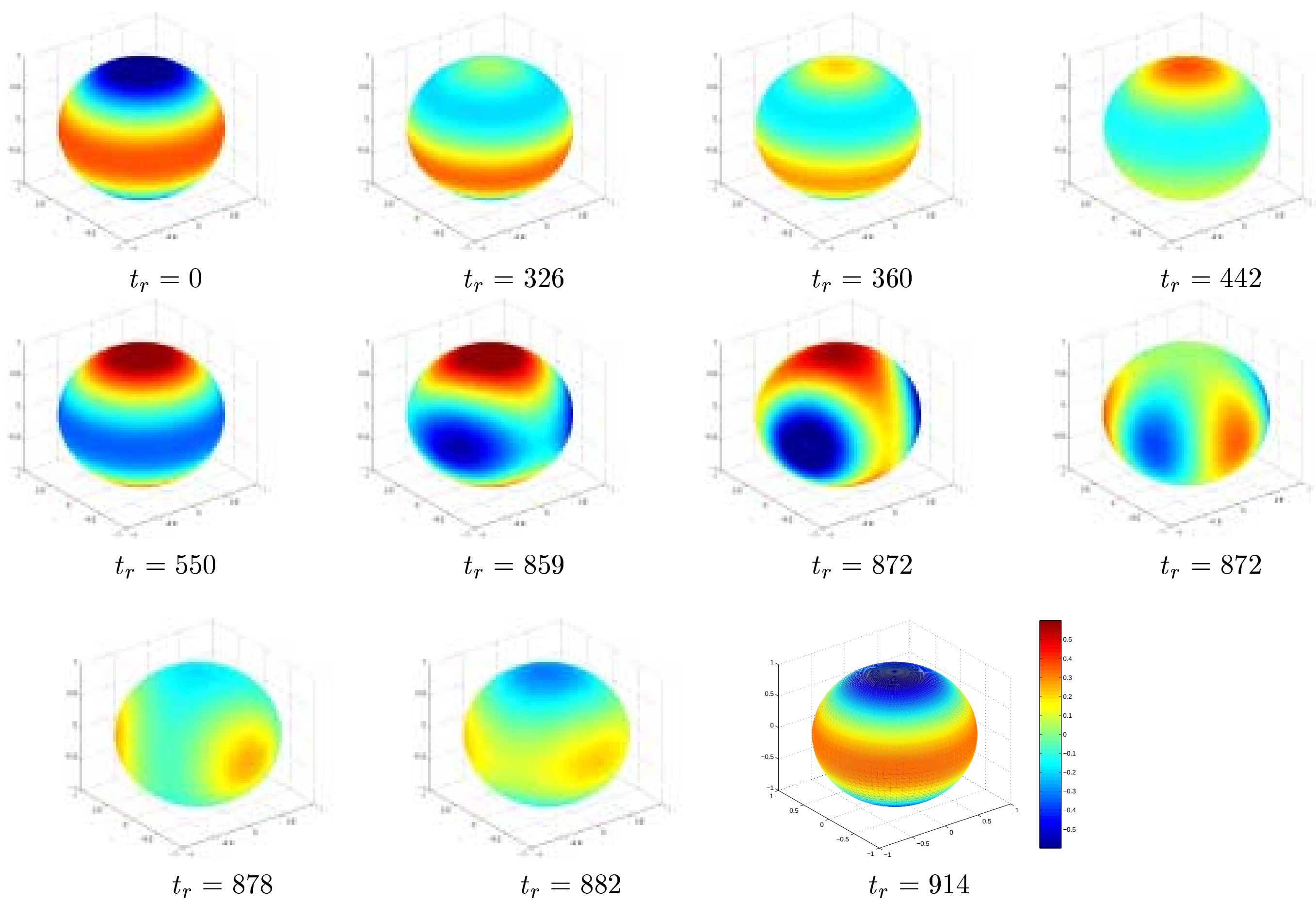

Figure 3: Temporal evolution of the radial velocity for a heteroclinic cycle ($\epsilon_\lambda = -0.04, \epsilon_\eta = -1.5$).