Finite Amplitude Subcritical Instability in Narrow-gap Spherical Couette Flow

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ABSTRACT

We consider the finite amplitude instability of incompressible spherical Couette flow between two concentric spheres of radii R_1 and R_2 $(> R_1)$ in the narrow gap limit, $\varepsilon \equiv (R_2 - R_1)/R_1 \ll 1$, caused by rotating them both about a common axis with distinct angular velocities Ω_1 and Ω_2 respectively. In this limit it is well known that the onset of (global) linear instability is manifest by Taylor vortices of roughly square cross-section close to the equator. According to linear theory this occurs at a critical Taylor number T_{crit} which exceeds the local value T_{cyl} obtained by approximating the spheres as cylinders in the vicinity of the equator, remarkably even as $\varepsilon \downarrow 0$, as shown in [1].

The weakly nonlinear extension of Soward and Jones' results [1] is not straightforward and so Harris *et al.* [2,3] focussed their attention on the case of almost co-rotation with $\delta \approx (\Omega_1 - \Omega_2)/\Omega_1 = O(\varepsilon^{1/2})$. In this limit the complex amplitude $\mathcal{A}(\chi, \tau)$ determining spatio-temporal modulation of the vortices is governed by the amplitude equation

$$\frac{\partial \mathcal{A}}{\partial \tau} = \left(\Lambda + 2\mathrm{i}\kappa\chi - \chi^2 - |\mathcal{A}|^2\right)\mathcal{A} + \frac{\partial^2 \mathcal{A}}{\partial \chi^2}\,, \tag{1}$$

in which $\Lambda \propto (T - T_{\text{cyl}})/\varepsilon$ and $\kappa \propto \delta/\varepsilon^{1/2}$ measure the excess Taylor number $T - T_{\text{cyl}}$ and angular velocity increment δ respectively, while τ is a suitably scaled time and $\chi \propto \theta/\varepsilon^{1/2}$ is a stretched latitude $-\theta$. The linearised version of (1) is locally unstable on the range $|\chi| < \Lambda^{1/2}$, on which the preferred frequency at given latitude is $2\kappa\chi$. Though the system is locally unstable whenever $\Lambda > 0$, the onset of (global) instability occurs when $\Lambda = \Lambda_{\text{crit}} = \kappa^2 + 1 = O(\kappa^2)$. This in part determines

$$T_{\text{crit}} = T_{\text{cyl}} + O(\delta^2) + O(\varepsilon)\,; \tag{2}$$

the $O(\varepsilon)$ correction is an estimate of higher order effects. The linear solution $\mathcal{A} \propto \exp(-\frac{1}{2}\chi^2 + \mathrm{i}\kappa\chi)$ at onset is localised near the equator over the latitudinal extent, $\theta = O(\varepsilon^{1/2})$, namely $\chi = O(1)$, while the vortex width $O(\varepsilon R_1)$ is modified by a small factor $O(\delta)$ because of the phase factor $\kappa\chi$. Curiously, when $\kappa \gg 1$, the unstable mode only occupies a small part of the locally unstable region of angular extent $O(\delta)$, namely $\chi = O(\kappa^{1/2})$.

The nonlinear version of (1) including the Stuart-Landau term [4] was first investigated by Hocking and Skiepko [5]. They identified the supercritical steady finite amplitude solution that follows the initial bifurcation and a secondary Hopf bifurcation to drifting phase solutions having the structure $\mathcal{A}(\chi, \tau) = \exp(\mathrm{i}\Omega\tau)\overline{a}(\chi)$. This forms the start of a rich bifurcation sequence unravelled in [2,3] by numerical integration of (1). Essentially for numerically large values of κ, typified by $\kappa = 4$, it was found that the steady state lost instability via a supercritical Hopf bifurcation to a vacilating solution. That expanded until it led to a gluing bifurcation, glued at the undisturbed state. Following the gluing bifurcation, waves travelling towards the equator were isolated characterised by a chevron pattern of space-time contours. These travelling wave solutions were subcritical and could be isolated for values of Λ far less than Λ_{crit}. Indeed the system typically underwent further bifurcations, which pointed to solutions exhibiting the underlying structure

$$\mathcal{A}(\chi, \tau) = \exp(\mathrm{i}\Omega\tau)\sum \overline{a}_n(\chi - \chi_n, \tau)\exp(\mathrm{i}n\omega\tau)\,, \tag{3a}$$

where each amplitude function $\overline{a}_n(\chi-\chi_n,\tau)$ is localised about

$$\chi \,=\, \chi_n \,=\, \tfrac{1}{2}(\Omega/\kappa) \,+\, n\,\Delta\chi\,, \qquad \text{separation length} \quad \Delta\chi \,=\, \tfrac{1}{2}(\omega/\kappa)\,, \tag{3b}$$

but with only a slow dependence on τ. Each term in the series (3a) determines a pulse which exhibits the local frequency $\Omega+n\omega$, and the sum determines a pulse train. For large κ the numerical results suggest the existence of such pulse trains when Λ is $O(\kappa^{2/3})$, small compared to the onset value $\Lambda_{\mathrm{crit}} = O(\kappa^2)$. It determines

$$T \,=\, T_{\mathrm{cyl}} \,+\, O((\delta\varepsilon)^{2/3}) \tag{4}$$

a value, which unlike (2) tends to the local cylinder value T_c as $\varepsilon\downarrow 0$. At such Taylor numbers, the latitudinal extent of local instability is $\theta = O((\delta\varepsilon)^{1/3})$, namely $\chi = O(\kappa^{1/3})$, while the pulse width is $\Delta\theta = O((\varepsilon^2/\delta)^{1/3})$ corresponding to $\Delta\chi = O(\kappa^{-1/3})$. The ratio of these lengths provides the estimate $O(\kappa^{2/3})$ for the number of pulses located in the locally unstable region.

All the calculations in [3] were undertaken with finite numerical values of κ. So to test the pulse train proposal, Bassom and Soward [6] developed an asymptotic theory based on the limit $\varepsilon\downarrow 0$ at fixed finite δ (corresponding to $\kappa\uparrow\infty$) with say $\Lambda=\kappa^{2/3}\widehat{\Lambda}$. Their strategy was to investigate the possibility of constructing pulse train solutions in the neighbourhood of some latitude $\theta = \theta_0 = O((\delta\varepsilon)^{1/3})$, corresponding to some $\chi=\kappa^{1/3}\widehat{\chi}_0$, on an intermediate angular scale large compared to the pulse width $O((\varepsilon^2/\delta)^{1/3})$ but small compared to the breadth $O((\delta\varepsilon)^{1/3})$ of the locally unstable region. On that intermediate scale we can approximate $\Lambda-\chi^2$ by $\kappa^{2/3}(\widehat{\Lambda}-\widehat{\chi}_0^2) = \kappa^{2/3}\widehat{\Lambda}_0$ a constant. Then upon writing $\mathcal{A}\propto a(x,t)\,\exp(2\mathrm{i}\kappa^{4/3}\widehat{\chi}_0\tau)$, where $x\propto-(\theta-\theta_0)$ and t are a rescaled local coordinate and time respectively, we arrive at the modified equation

$$\frac{\partial a}{\partial t} \,=\, \big(\lambda \,+\, \mathrm{i}x \,-\, |a|^2\big)\,a \,+\, \frac{\partial^2 a}{\partial x^2}\,, \tag{5}$$

in which $\lambda\propto\widehat{\Lambda}_0$.

The steady version of (5) with $\partial a/\partial t = 0$ has curious properties. Its linearised version certainly has no localised solutions and all previous attempts to find nonlinear solutions with $a\to 0$ as $|x|\to\infty$ have failed (Hocking private communication *circa*. 1980, and [7]). The possibility of pulse trains was proposed for a related system in [8] but was not explored in the context of (5), as it was erroneously believed that their non-existence would be a consequence of the absence of isolated pulses. This negative view was cast into doubt by the evidence in [3]. So solutions of initial value problems with the property that $\exp(-\mathrm{i}xt)\,a$ is spatially periodic in x were sought numerically using Fourier transform methods developed in [8]. Following the decay of transients robust pulse trains were identified with the structure

$$\begin{aligned} a(x,t;L) \,=\, & \mathrm{e}^{\mathrm{i}\pi/4}\sum_{\forall n}\exp\left[\mathrm{i}\left(2n+\tfrac{1}{2}\right)Lt\right]\overline{a}(x-\left(2n+\tfrac{1}{2}\right)L) \\ & + \mathrm{e}^{-\mathrm{i}\pi/4}\sum_{\forall n}\exp\left[-\mathrm{i}\left(2n+\tfrac{1}{2}\right)Lt\right]\overline{a}(x+\left(2n+\tfrac{1}{2}\right)L)\,, \end{aligned} \tag{7}$$

parameterised by the constant L which measures the distance between the pulse centres. The function $\overline{a}(x)\,(=\overline{a}^*(-x)$, where the star denotes the complex conjugate) characterising each pulse solves

$$\frac{\mathrm{d}^2\overline{a}}{\mathrm{d}x^2} \,+\, (\lambda+\mathrm{i}x)\,\overline{a} \,=\, \sum_{\forall m,n}\sigma_{m,n}\,\overline{a}(x-mL)\,\overline{a}(x-nL)\,\overline{a}^*(x-(m+n)L)\,, \tag{8a}$$

where, for integer m, n,

$$\sigma_{m,n} \,=\, \begin{cases} -1\,, & m \text{ and } n \text{ are both odd,} \\ 1\,, & \text{otherwise,} \end{cases} \tag{8b}$$

subject to $\overline{a} \to 0$ as $|x| \to \infty$. The phase shifts between neighbouring pulses evident in the structure of (7) are crucial and lead to the curious coefficients $\sigma_{m,n}$ in (8a). Without the phase shifts all the $\sigma_{m,n}$ would be unity, and in that simpler case we continue to believe that no solution exists as in the steady case. Herein lies the success of the new results. We must stress however that such solutions exist over a finite range $L_{\min}(\lambda) \le L \le L_{\max}(\lambda)$ dependant on λ, provided that $\lambda > \lambda_{\inf} \approx 2.54074$, where $L_{\min}(\lambda_{\inf}) = L_{\max}(\lambda_{\inf}) \equiv L_{\inf} \approx 2.11831$.

On the one hand, the pulse train solutions appear to be robust in the sense that our time stepped numerical solution with given periodicity length L locked on to them rapidly. On the other, we compared our predictions with the $\kappa = 4$ results obtained by Harris *et al.* [3] and found excellent qualitative and quantitative agreement despite the fact that κ was only moderately large. For example at $\lambda = 4$ we found that the pulse mean energy

$$\langle \mathcal{E} \rangle \equiv \frac{1}{2\pi} \int_{-T}^{T} \int_{-L/2}^{L/2} |a(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{L} \int_{-\infty}^{\infty} |\overline{a}(x)|^2 \, \mathrm{d}x \,, \tag{9}$$

where $T = \pi/L$ (temporal period $4T$), is maximised over $L_{\min}(4)[\approx 1.26] \le L \le L_{\max}(4)[\approx 2.77]$ at $L \approx 2.35$. For a well-studied case with $\lambda = 3.875$ at the equator $\theta = 0$, Harris *et al.* [3] obtained $L \approx 2.42$.

From the point of view of (1), the pulse train solution (7) must be slowly modulated as λ is a function of latitude. Nevertheless, we may keep L fixed as suggested by (3). Significantly, as λ decreases to zero at the edge of the locally unstable region the pulse train amplitude decreases to its minimum (but finite) value when λ satisfies $L_{\max}(\lambda) = L$. Beyond that latitude no pulses are possible and so the vortex amplitude collapses to zero over the pulse width L. Evidently our theory cannot be applied at the edge of the pulse train. There conditions are delicate, which probably explains why the long time behaviour of Harris *et al.*'s solutions generally exhibited complex behaviour. For some (but not all) equator values of λ a beating frequency could be identified.

Interestingly, however, in addition to the physical processes captured by (1), we identify a group velocity c_g proportional to the latitude $-\theta$ directed away from the equator and of magnitude $O((\delta/\varepsilon)|\theta|\nu/R_1)$, where ν is the viscosity. This is of exactly the same magnitude of the phase (i.e. the drift) velocity c_p of the Taylor vortices which is also proportional to the latitude $-\theta$ but directed towards the equator. Thus the pulses themselves drift outwards at the group velocity. In consequence their separation L increases with time, fortunately at a rate independent of position, so that the pulse train structure can remain spatially uniform. Nevertheless, there are necessarily long time repercussions. For with L increasing indefinitely, the train must either eventually collapse or undergo instability. Our belief is that, due to the robust nature of the pulse trains, they will reemerge with L values which tend to maximise pulse energy.

Our physical picture is as follows. At a given latitude $-\theta$ there is a natural Taylor vortex frequency $O((\delta/\varepsilon^2)|\theta|\nu/R_1^2)$. However, the realised value rather than varying continuously retains a constant value over the pulse width $O((\varepsilon^2/\delta)^{1/3}R_1)$. Under the pulse all Taylor vortices, width εR_1, propagate towards the equator at the same speed, the phase speed c_p based on that frequency. In view of the uniformity of the pulse separation L, the frequency increment between any two neighbouring pulses is the same $O((\delta/\varepsilon^2)^{2/3}(\nu/R_1^2))$ and this enables each pulse to interact coherently with its neighbours. Significantly, the frequency increment determines a corresponding phase speed increment. This means that the space-time chevron pattern for the vortices will exhibit dislocations half way between the pulse centres at the point where one pulse looses dominance to its neighbour (i.e. the inclination of the wave fronts to the $\theta = 0$ axis, which are constant for each pulse, decrease to a shallower angle on moving away from the equator from one pulse to the next).

Evidently the link with both laboratory and numerical experiments is somewhat tenuous. The main difficulty faced is that we have identified behaviours on three different length and time scales, which are most readily appreciated for the important $\delta = 1$ case corresponding to the situation when the outer sphere is at rest. In summary, there is the short length scale εR_1 and time scale $\varepsilon^2 R_1^2/\nu$ of the vortices. Second, there is modulation on the intermediate length scale $\varepsilon^{2/3} R_1$ of the pulses for which the relevant time scale $\varepsilon^{4/3} R_1^2/\nu$ is inversely proportional to the frequency increment between neighbouring pulses. This is the space-time range over which our analysis is valid. Third, the pulses exist and are spatially modulated on a relatively wide locally unstable region width $O(\varepsilon^{1/3} R_1)$, though this is still short compared to the $O(R_1)$ length associated with the distance between the pole and the equator. The long time associated with temporal modulation caused by pulse separation due to the group velocity is $O(\varepsilon^{4/3} R_1^2/\nu)$. We can only speculate on the complicated spatio-temporal evolution over these longest scales. It would be difficult to conduct experiments at a sufficiently small ε such that these scale separations can be distinguished.

Though we have not proved that the pulse-trains persist on the longest time scales we have shown how the basic pulse unit can support the existence of its neighbour. Indeed the essential idea is that at every location there is a preferred frequency which increases linearly with respect to distance from the equator. Moreover the initial value calculation for spatially periodic solutions was formulated with the factor $\exp(\mathrm{i}tx)$, as explained below (5), to accommodate that preference. Nevertheless the realised temporally periodic forms to which the solution settles after the transients decay possess the discrete set rather than a continuous distribution of frequencies. Each pulse is localised in the vicinity of the point at which the frequency is preferred. Furthermore, the constant frequency jump between neighbouring pulses is essential for their mutual resonance. The fact that the pulse-train solutions emerge naturally as the solution to an initial value problem suggests that they are robust. Before the calculations reported in [6] were undertaken, it was far from clear whether pulse-trains were even possible. Our demonstration of their existence provides an affirmative answer to the long outstanding question as to whether subcritical finite amplitude solutions can occur in the vicinity of the local critical cylinder Taylor number T_{cyl}.

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