

The Asymptotic Structure of Long–Wave Görtler Vortices in a Hypersonic Boundary Layer

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ABSTRACT

A hypersonic boundary layer on a concave surface is considered. It is supposed, that a non–dimensional surface curvature is small $k \ll 1$, a gas is perfect, a viscosity μ has the linear dependence upon an enthalpy h and pressure perturbations due to the boundary layer displacement thickness and the surface curvature effect are small in comparison with the free stream pressure value. Then a boundary layer with the characteristic thickness $\Delta y \sim \delta \sim M_\infty^2 / Re_\infty^{1/2}$ is described by the self–similar equations

$$f''' + ff'' = 0, \quad \frac{h''}{Pr} + fh' + (f'')^2 = 0 \quad (1)$$

$$f(0) = f'(0) = 0, \quad h(0) = h_w \quad \text{or} \quad h'(0) = 0, \quad f'(\infty) = 1, \quad h(\infty) = \frac{1}{(\gamma - 1)M_\infty^2}$$

Here $M_\infty \gg 1$ is the free stream Mach number, $Re_\infty = \rho_\infty u_\infty L / \mu_\infty \gg 1$ is the Reynolds number, Pr is the Prandtl number and γ is the specific heat ratio.

It is known that two–dimensional laminar boundary layer on a concave surface can lose stability if the Görtler number $G_\infty = 2kRe_\infty^{1/2}/M_\infty^2$ exceeds some critical value. Then extended streamwise steady Görtler vortices are formed in a boundary layer. Below, the vortices development is investigated when their wavelength exceeds a boundary layer thickness and the Görtler number is large $G_\infty \sim k/\delta \gg 1$.

It is supposed that the vortices occupy all boundary layer thickness and their formation generates non–linear disturbances (for example, for the longitudinal velocity $\Delta u \sim u \sim 1$). Then it is obtained from the comparison of orders of the Navier–Stokes equations convective terms that in the vortical region with the thickness $\Delta y \sim \delta$ in a centrifugal flow field there is the additional pressure perturbation $\Delta p \sim k\delta/M_\infty^2$, which induces the spanwise velocity $w \sim (k\delta)^{1/2}$. Estimates for the vortical region spanwise scale $\Delta z \sim (k\delta)^{1/2}\Delta x$ and the vertical velocity $v \sim \delta/\Delta x$ are obtained from the discontinuity equation, here $(\delta/k)^{1/2} \leq \Delta x \leq 1$ is the longitudinal scale of the vortical region. If $\Delta x \sim 1$ we can obtain the estimate for the vortical region maximal spanwise scale or the maximum vortex wavelength in gas – $\Delta z_{max} \sim (k\delta)^{1/2}$. The upper estimates show also that the interaction between the vortical region and an external oncoming flow is absent. Therefore for the vortical region with the characteristic scales

$$\Delta y \sim \delta, \quad \Delta z \sim (k\delta)^{1/2}\Delta x, \quad \left(\frac{\delta}{k}\right)^{1/2} \leq \Delta x \leq 1, \quad \Delta y \leq \Delta z \ll \Delta x$$

it is possible to formulate boundary–value problem for the nonlinear development of long–wave Görtler vortices

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \quad \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \frac{\Delta x}{Re} \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (2)$$

$$\begin{aligned}
& \left(\frac{\delta}{k}\right) \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) + \Delta x^2 \lambda^2 \left(\rho u^2 + \frac{\partial p}{\partial y} \right) = 0 \\
& \rho \left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = \frac{\Delta x}{Re} \frac{\partial}{\partial y} \left(\mu \frac{\partial w}{\partial y} \right) \\
& \rho \left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \frac{\partial h}{\partial z} \right) = \frac{\Delta x}{Re} \left[\frac{1}{Pr} \frac{\partial}{\partial y} \left(\mu \frac{\partial h}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 \right], \quad \rho h = 1, \quad \mu = h \\
& u = v = w = 0, \quad h = h_w \quad \text{or} \quad \frac{\partial h}{\partial y} = 0 \quad (y = 0); \quad u \rightarrow 1, \quad w \rightarrow 0, \quad h \rightarrow \frac{1}{(\gamma - 1)M_\infty^2} \quad (y \rightarrow \infty) \\
& u = u_0(y), \quad v = \frac{\Delta x}{Re} v_0(y), \quad w = 0, \quad p = - \int_0^y \rho_0 u_0^2 dy, \quad \rho = \rho_0(y), \quad \mu = \mu_0(y), \quad h = h_0(y) \quad (x = 0) \\
& f(x, y, z) = f(x, y, z + 2\pi), \quad f = u, v, w, p, \rho, h, \mu
\end{aligned}$$

where $Re \sim 1$ is the local Reynolds number, $\lambda \geq 1$ is referred to a boundary layer thickness vortex wavelength and the index "0" corresponds to the function profiles in the undisturbed boundary layer (1) in some section $x_0 \sim 1$.

If the longitudinal scale of the vortical region is small $\Delta x \sim (\delta/k)^{1/2} \ll 1$ the spanwise scale (or the vortex wavelength) is asymptotically equal to a boundary layer thickness $\Delta z \sim \Delta y \sim \delta$. In this case the vortices evolution will occur in the one-dimensional parallel flow and viscous terms will negligible in the equations (2). The subsequent linearization, the solution normal-mode approach using

$$F(x, y, z) = \bar{F}(y) \exp(\beta x) (\sin z, \cos z)$$

and the boundary layer vertical coordinate introduction will transform them to the ordinary differential equation for function $V_1 = V/u_0$ where V is the vertical velocity

$$V_1'' + 2 \left(\frac{u_0'}{u_0} - \frac{h_0'}{h_0} \right) V_1' - \left(\frac{h_0}{\Lambda} \right)^2 V_1 = \left(h_0' - 2 \frac{u_0' h_0}{u_0} \right) \frac{V_1}{B^2} \quad (3)$$

$$V_1'(0) = V_1(\infty) = 0, \quad \Lambda = \frac{\lambda}{(2x_0)^{1/2}}, \quad B = \frac{\beta}{(2x_0)^{1/4}}$$

The asymptotic structure of such vortices was considered also in [1–3].

The eigen-value problem solution (3) has shown that the first mode increment B_1 is increased with the Mach number and the relative vortex wavelength Λ growth, but the higher mode increments are decrease with M_∞ growth and do not depend on Λ practically. Such property of increment B_1 change means that the first mode is separated from the higher ones when the vortex wavelength increase and the linear development have to occur on the smaller characteristic scale Δx .

An increasing of the Mach number leads to a boundary layer heating with increasing of its thickness $\delta \sim M_\infty^{3/2} \delta_f$ and the vortical region longitudinal scale $\Delta x \sim M_\infty^{3/4} \Delta x_f$ in comparison with the values for a fluid boundary layer. It causes the reduction in the vortex growth rate (referred to a characteristic length of the order of unity)

$$Be \sim \frac{B}{\Delta x} \sim \frac{B}{M_\infty^{3/4} \Delta x_f}$$

as in this fraction the denominator change is the main.

It is obtained that the surface heating from a strongly cooled surface to an adiabatic one increases a little the increment B_1 but the higher mode increments are increased approximately twice. However, the surface heating increases the boundary layer thickness and the vortical region scales also. Therefore it is unable to estimate the surface heating influence on the vortex growth rate Be .

It is obtained that the Prandtl number Pr increasing raises a little the value B_1 and does not change the higher mode increments practically.

It is found that the vortex growth rate $Be \sim B/\Delta x \sim B/\Lambda$ decreases with Λ growth, as in this fraction the denominator change is the main also.

It is shown from the eigenfunction profiles V that with Λ increasing their attenuation occurs on the increasing distances from a surface.

It is considered now the vortices development when they induce only small disturbances ($\Delta u \ll u \sim 1$, for example) in the boundary layer main part with the thickness $\Delta y \sim \delta$ and nonlinear disturbances ($\Delta u \sim u \ll 1$) in its near-wall part with the thickness $\Delta y/\delta \ll 1$. Assuming that the friction–stress and the heat–flux preserve their orders of magnitude in the boundary layer near-wall part it is possible to get the profiles of the longitudinal velocity u and the enthalpy h

$$u \approx \frac{y}{\delta h_w}, \quad h \approx h_w + \frac{y}{\delta h_w} \quad \text{at} \quad \left(\frac{y}{\delta}\right)^{1/2} \ll h_w \leq 1 \quad (4)$$

Supposing, that the flow in the vortical region near wall part (region 3) is viscous, three–dimensional and nonlinear it is possible to obtain from the Navier–Stokes equations the estimates for its thickness $\Delta y \sim h_w \delta \Delta x^{1/3}$ and the pressure perturbation $\Delta p \sim \Delta z^2/M_\infty^2 h_w \Delta x^{4/3}$.

In the vortical region main part (region 2) with the thickness $\Delta y \sim \delta$ the pressure perturbation is created by the centrifugal effects $\Delta p \sim k \delta \Delta u/M_\infty^2$. As it should have the same order of magnitude as in the region 3 it is possible to obtain estimates for the longitudinal velocity perturbation $\Delta u \sim \Delta z^2/h_w \Delta x^{4/3} k \delta$ and for the vertical velocity $v \sim \Delta z^2/h_w \Delta x^{7/3} k$.

The pressure perturbation grows in the order of magnitude in region 2 and on its outer edge is $\Delta p \sim \Delta z^2/h_w \Delta x^{4/3}$. Such pressure perturbation induces the vertical velocity $v \sim \Delta z/h_w \Delta x^{1/3}$ in the vortical region outer part (region 1) with the thickness $\Delta y \sim \Delta z \gg \delta$ (it is the disturbed part of an uniform oncoming flow).

The obtained estimates show, that the interaction of the regions 3 and 2 or the regions 2 and 1 is realized at $\Delta z \sim h_w k^{1/2} \delta^{1/2} \Delta x^{5/6}$ or at $\Delta z \sim k \Delta x^2$. When these conditions are carried out simultaneously it is that $\Delta x \sim h_w^{6/7} (\delta/k)^{3/7}$, $\Delta z \sim h_w^{12/7} k^{1/7} \delta^{6/7}$ and the triple–deck structure with an interaction of the vortical flow is realized. Then the next boundary–value problem is obtained for the region 3

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0 \\ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} &= \frac{\partial^2 w}{\partial y^2} \\ u = v = w = 0 \quad (y = 0); \quad u \rightarrow y + \gamma_1 D, \quad w \rightarrow 0 \quad (y \rightarrow \infty) \\ u \rightarrow y, \quad v, w, p, D \rightarrow 0 \quad (x \rightarrow -\infty); \quad p = \gamma_2 P(Y = 0) + D \\ u, v, w(x, y, z) = u, v, w(x, y, z + 2\pi); \quad p, D(x, z) = p, D(x, z + 2\pi) \end{aligned} \quad (5)$$

where the parameters γ_1 and γ_2 define an interaction measure of the regions 3 and 2 and the regions 2 and 1 accordingly. The boundary layer edge is moved on the region 3 displacement thickness $D(x, z)$ and it induces the pressure distribution $P(x, Y, z)$, which is determined from the solution of a wave boundary–value problem for region 1

$$\gamma_3 \frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 P}{\partial Y^2} + \frac{\partial^2 P}{\partial z^2}, \quad \frac{\partial P}{\partial Y} = \frac{\partial^2 D}{\partial x^2} \quad (Y = 0); \quad P(x, Y, z) = P(x, Y, z + 2\pi) \quad (6)$$

where the parameter γ_3 defines the property of wave disturbances propagation. Subsequent boundary–value problem (5) linearization and using the normal–mode solution representations allows to reduce it and (6) to the ordinary differential equations system, for which it is possible to get a dispersion ratio

$$\frac{\gamma_2 \beta^2}{(1 + \gamma_3 \beta^2)^{1/2}} - 3\gamma_1 Ai'(0) \beta^{5/3} = 1$$

It differs from the appropriate expression for a fluid [4,5] only by the parameter γ_3 presence. The estimates show, that $\gamma_3 \sim h_w^{12/7}$. Therefore at surface cooling the wave disturbances extending (6) accepts the fluid character, the dispersion ratio is transformed to the fluid kind, the increment β does not depend any more on h_w and the transformed to characteristic length ~ 1 vortex growth rate

$$Be \sim \frac{\beta}{\Delta x} \sim \left(\frac{k}{\delta}\right)^{3/7} \frac{\beta}{h_w^{6/7}}$$

is increased due to the surface enthalpy h_w and the boundary layer thickness δ reduction. Thus it is analytically shown, that the surface cooling increases the long–wave Görtler vortex growth rate.

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