Complex dynamics in a short Taylor-Couette annulus

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ABSTRACT

Recent experimental results (Mullin & Blohm 2001) have revealed very interesting dynamics of the flow in a short Taylor-Couette annulus where the top endwall and outer cylinder are stationary and the flow is driven by the constant rotation of the inner cylinder and bottom endwall. This arrangement results in a system with $SO(2)$ as the only symmetry (invariance to rotations about the axis).

We consider an incompressible flow confined in an annulus of inner radius R_i and outer radius R_o and length L, driven by the constant rotation of the inner cylinder and bottom endwall at Ω rad/s while the outer cylinder and top endwall remain at rest. The system is nondimensionalized using the gap, $D = R_o - R_i$, as the length scale and the diffusive time across the gap, D^2/ν, as the time scale (where ν is the fluid's kinematic viscosity). The equations governing the flow are the Navier-Stokes equations together with initial and boundary conditions. Keeping the radius ratio fixed at $\eta = R_i/R_o = 0.5$, we consider the dynamics as the other two governing parameters are varied. These parameters are

$$\text{Reynolds number:} \quad Re = \Omega D R_i/\nu,$$
$$\text{annulus aspect ratio:} \quad \Gamma = L/D.$$

A schematic of the flow geometry, with an insert of the streamlines for a three-cell steady axisymmetric solution at $Re = 124.5$, $\Gamma = 3.10$ is shown in figure 1.

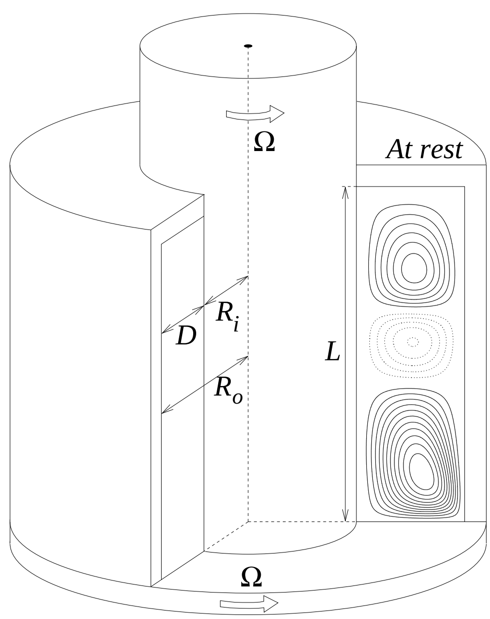

Figure 1: Schematic of the flow geometry, with an insert of the streamlines (solid are positive and dashed are negative contours of the streamfunction) in an (r, z) meridional section for a three-cell steady axisymmetric solution at $Re = 124.5$ and $\Gamma = 3.10$.

Z_2 symmetry is a very important symmetry in Taylor-Couette flow with endwalls; the symmetry corresponds to invariance to reflection about the annulus half-height. It influences many of the early bifurcations from the basic state. A recent experimental study [1] has looked at primary instabilities in a Taylor-Couette system that lacks Z_2 symmetry. Z_2 symmetry was removed by having one endwall rotate with the inner cylinder and the other endwall remain stationary along with the outer cylinder. In that study, the focus was on the case of a relatively short annulus where there are states with either one or three cells, A_1 and A_3 states, respectively, and a regime where these two states compete. The primary competition between the two states results in hysteretic behavior as parameters are varied. In the experiments, the radius ratio of the cylinders was held fixed and only two parameters were varied, the annulus aspect ratio and the inner cylinder rotation rate. In this two-parameter space, the hysteresis manifests itself as a pair of saddle-node (fold) bifurcation curves which meet at a codimension-2 cusp point. The associated bifurcations were found to be steady and axisymmetric, and excellent agreement between nonlinear steady axisymmetric computations and the experiments was observed. Figure 2 is a reproduction of their figure 7, summarizing their findings.

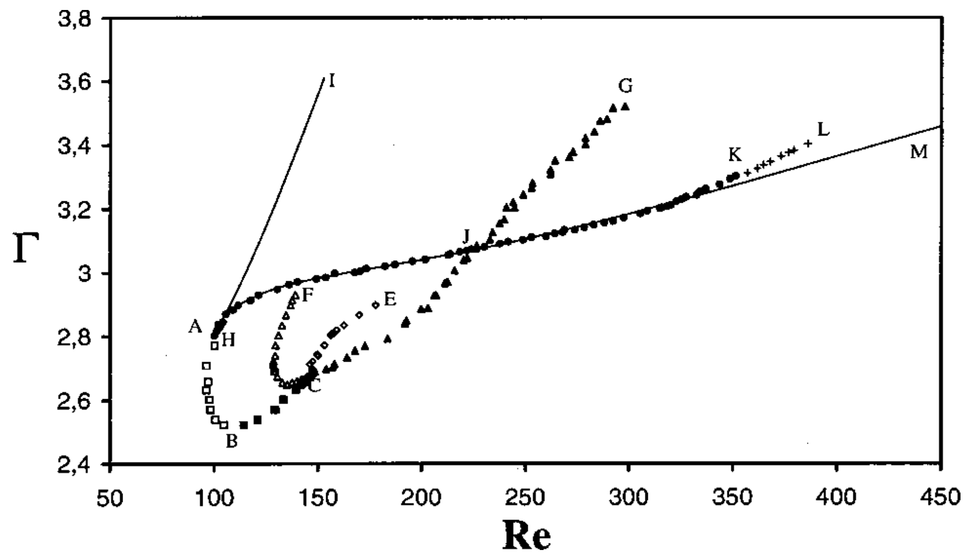

Figure 2: Reproduction of figure 7 from [1]; the curve HL is the path of saddle-nodes for three-cells, HI that for single-cell, and H corresponds to a cusp point; the curves ABC and CG correspond to two different Hopf bifurcations, the point C is the double Hopf bifurcation point. The curves CE and CF are curves at which one or other of the time periodic states could not be followed across. The symbols represent experimentally determined points, and the solid curves HM and HI were numerically determined.

The experiments of [1] also revealed interesting time-dependent behavior in which the $SO(2)$ symmetry (invariance to arbitrary rotations about the annulus axis) was broken via supercritical Hopf bifurcations. They also documented dynamics associated with a double Hopf bifurcation. The resulting non-axisymmetric time-periodic states were beyond the capabilities of their numerics, and many open questions remained, such as what is the spatial structure of the three-dimensional states, and what happens to these in the parameter regimes where the A_1 and A_3 states compete, i.e., in the hysteretic (fold) region. We have used a three-dimensional Navier-Stokes solver to address these questions and to further explore the nonlinear dynamics associated with the observed double Hopf bifurcation. Mullin & Blohm (2001) did not report on the specific nature of the unsteady states (other than to give the frequency and amplitude of the periodic variations in the radial velocity at a point). Our numerics have revealed that the two Hopf bifurcations are symmetry-breaking to rotating waves with azimuthal wave numbers one

(RW$_1$) and two (RW$_2$). Although the spatial wavenumbers are in a 1:2 ratio, the double Hopf bifurcation is non-resonant as the associated precession frequencies of the rotating waves at the bifurcation (i.e., the two critical pairs of complex conjugate eigenvalues) are not in a 1:2 ratio. Hence the double Hopf bifurcation, even though it is taking place in an $SO(2)$-equivariant system, has the generic normal form (see the Appendix in Marques, Lopez & Shen 2002 for details). Figure 3 sumarizes the results obtained in the numerical simulations.

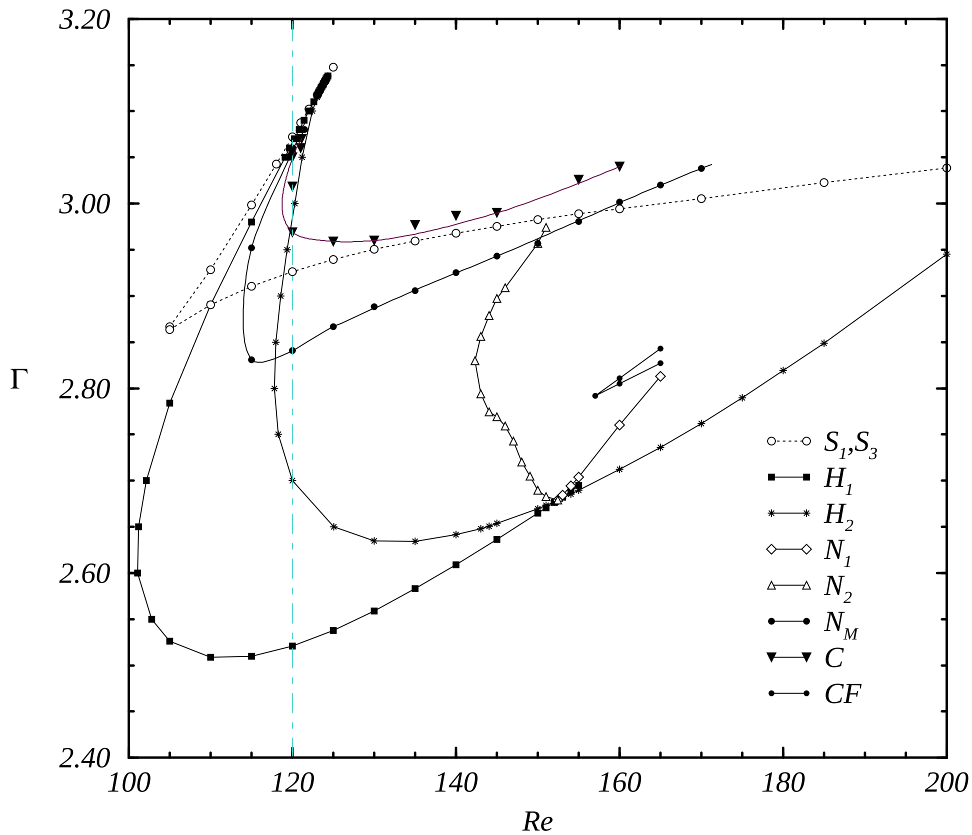

Figure 3: Loci in (Re, Γ)-space of the bifurcation curves described in the text.

The dynamics were shown to be organized by a pair of codimension-2 bifurcations: a cusp bifurcation where two curves of (axisymmetric and steady) saddle-node bifurcations (S_1, S_3) meet, and a double Hopf bifurcation where two Hopf bifurcation curves (H_1, H_2) intersect. The Hopf bifurcations were observed to be supercritical and both broke the $SO(2)$ symmetry resulting in three-dimensional time-periodic states. Coexistence of both stable rotating waves (RW$_1$, RW$_2$) happens in the region bounded by Neimark-Sacker bifurcations curves (N_1, N_2 in figure 3). Our computations reproduce all of these dynamics, and further identify that the two Hopf bifurcations result in rotating wave states with azimuthal wavenumbers 1 and 2, respectively. The computed precession frequencies agree very well with the experimentally measured frequencies (obtained using laser Doppler velocimetry at a point). Even the curves of secondary Hopf bifurcations (Neimark-Sacker bifurcations) associated with the double Hopf bifurcation are determined numerically and found to agree very well with the experimentally determined curves.

The numerical computations have also allowed a detailed exploration of the flow dynamics in the fold region associated with the cusp bifurcation. In this region, we have found that a pair of fold-Hopf bifurcations (codim-2 points, where curves saddle-node and Hopf bifurcations intersect tangentially). At

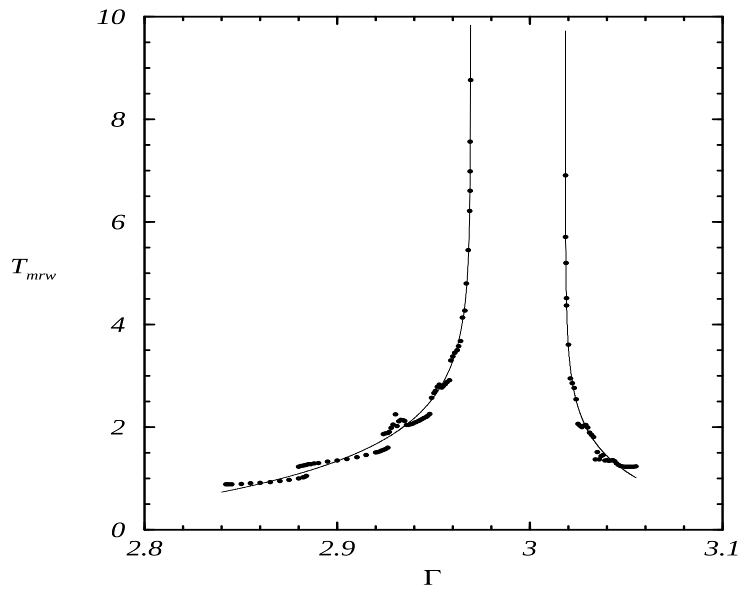

Figure 4: Modulation period of the MRW solutions for $Re = 120$.

least one of these is of the complicated type where a Neimark-Sacker bifurcation curve N_M and a thin horn-shaped region of complicated dynamics (involving sequences of saddle-nodes, period-doublings, and heteroclinic and homoclinic collisions) are spawned; a curve of homoclinic/heteroclinic collision C between a modulated rotating wave (resulting from the Hopf instability of a rotating wave) and saddle equilibria (either A_1 and/or A_m) in the fold of the cusp bifurcation is determined numerically. All of the associated dynamics are detected numerically, and a detailed bifurcation diagram is obtained that consistently shows the inter-connections between the dynamics associated with the codimension-2 bifurcation points (cusp, double Hopf and fold-Hopf points) and accounts for all the complicated dynamics in a extensive region of parameter space.

In figure 4, the period of the modulated rotating waves T_{MRW} when we approach the homoclinic/heteroclinic curve C, as a function of Γ is shown (vertical dashed line in figure 3). The filled circles in the figure are values of T_{MRW} determined from individual computational cases, and the solid lines are logarithmic fits. The complex behavior before the collision, reminiscent of Silnikov phenomena, is clearly observed.

Cyclic-fold bifurcations (saddle-node bifurcations for limit cycles) have also been found in the region of coexistence of both stable rotating waves (RW_1, RW_2). These cyclic-fold bifurcations CF colides in a cusp bifurcation of limit cycles. We believe that these phenomena are novel in fluid dynamics, as is a detailed exploration of how their associated dynamics are all inter-connected.

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REFERENCES

[1] T. Mullin & C. Blohm *Bifurcation phenomena in a Taylor-Couette flow with asymmetric boundary conditions*, Phys. Fluids **13** (1), 136–140, 2001.

[2] F. Marques, J.M. Lopez & J. Shen *Mode interactions in an enclosed swirling flow: A double Hopf bifurcation between azimuthal wavenumbers 0 and 2*, J. Fluid Mech. **455**, 263–281, 2002.