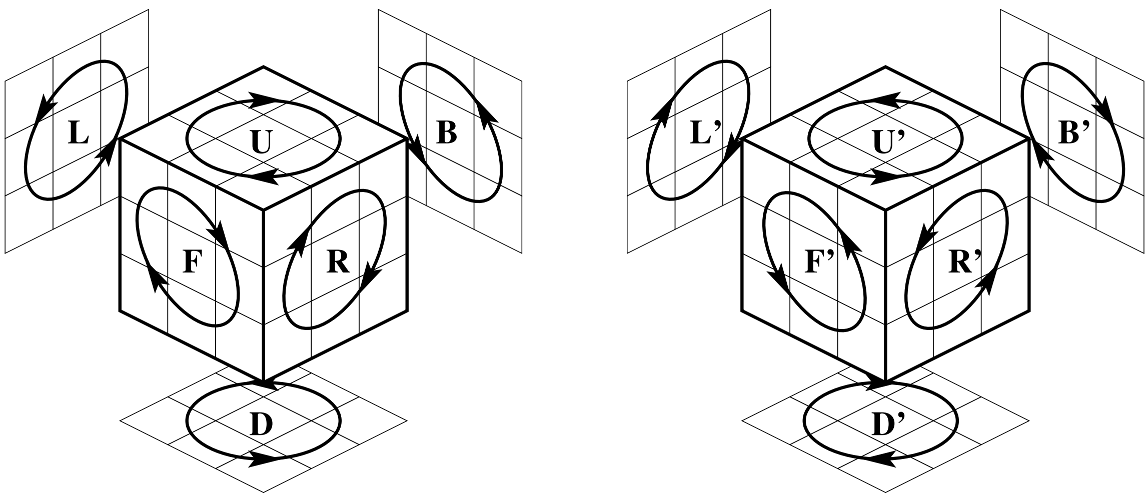
Solving the Rubik's cube

1 Notation

The cube has six faces: up (U), down (D), front (F), back (B), left (L) and right (R). Every face can be independently rotated. We use the same symbols U, D, F, B, L and R for the rotation of the corresponding face 90° (i.e. a quarter turn) clockwise (looking from outside the cube!). The opposite rotations (90° counterclockwise) of the faces are denoted U', D', F', B', L' and R', as indicated in the figure.



Every movement can now be described as a sequence of rotations. When doing a complex movement, DO NOT CHANGE the cube ORIENTATION, which means that the CENTER CUBES of all six faces MUST NOT BE MOVED, pointing always in the same direction (up, down, front, left or right).

A half turn of a face is made of two 90° rotations. For the top face, this is the movement RR, and we will write it as R^2 for convenience. Analogously, RRR = R^3 = R'; making four 90° rotations, RRRR = R^4, or one rotation and its reverse, RR' = R'R, is the same as *doing nothing*.

A very important movement is the *commutator* of A and C, written [A,C], and defined as

$$[A,C] = ACA'C'$$

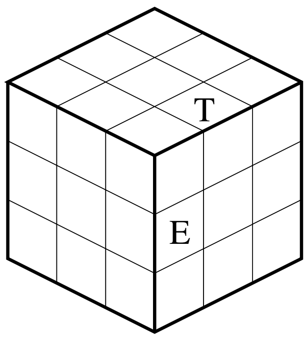
where A and C are arbitrary movements (U, D, F, B, L, R, their reverses, or any other combination).

The Rubik's cube can be solved just by a careful analysis of the commutator.

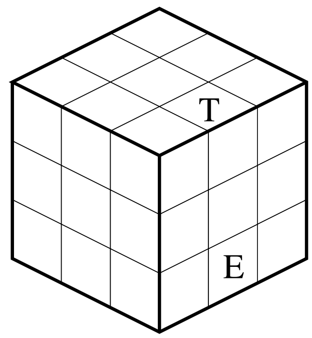
2 Solving the bottom slice

2.1 Putting edges in place

The edges are the cubes with two colored faces. First we must put the four edges that belongs to the bottom slide on the top slide.



R

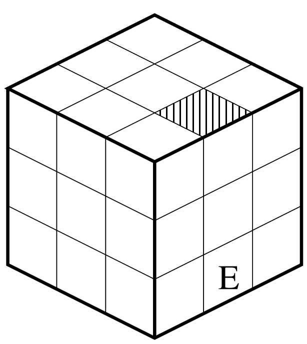


R^2

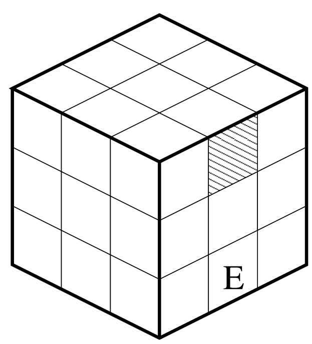
If a bottom edge E is on the middle slide (left figure) then rotate the upper slide until the cube T is NOT an edge of the bottom slide, and apply the move R.

If a bottom edge E is on the bottom slide (left figure), but it is not in place or it is flipped, then rotate the upper slide until the cube T is NOT an edge of the bottom slide, and apply the move R^2.

Second, we move the four edges on the upper slide on its place on the bottom slide:



R^2

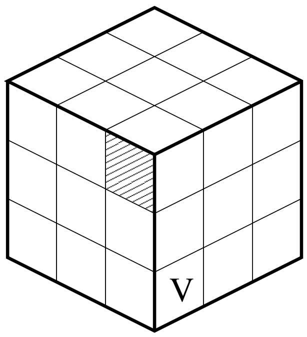


UFR'F'

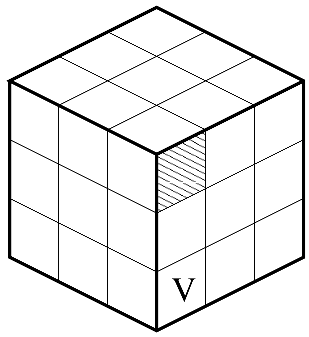
The edge that should be at E is on the top slide, and by rotating the top face we put it facing E; the bottom face color may be on the top face or the right face (the hatched face of the edge), and these are the two cases in the figure.

2.2 Putting corners in place

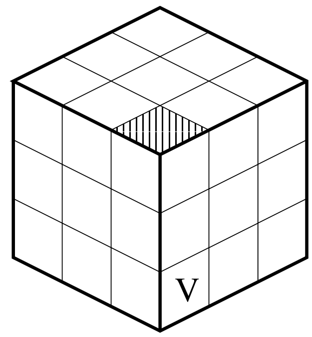
The corners are the cubes with three colored faces. Assume that the corner of the bottom slide that should be on V position is on the upper slide; turn the upper slide until it faces V; the bottom face color can be in three different positions (the hatched face of the corner):



URU'R'



U'F'UF

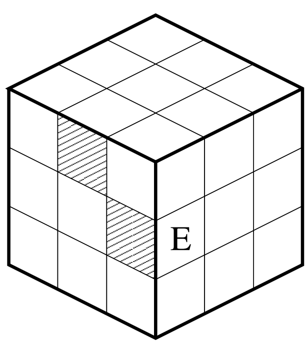


F'UFU'RU'R'

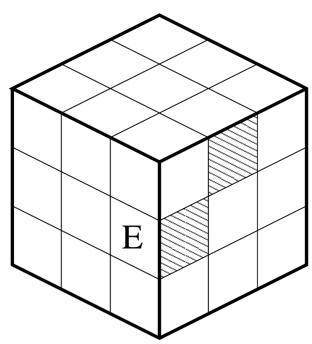
If a corner of the bottom slide is on the bottom slide but out of place or rotated, we just replace it with any of the corners of the top slide with the first move in the previous figure; this will bring the corner to the upper slide, that we have just learnt how to solve.

3 Solving the middle slice

There are two cases here:



URU'R'.U'F'UF



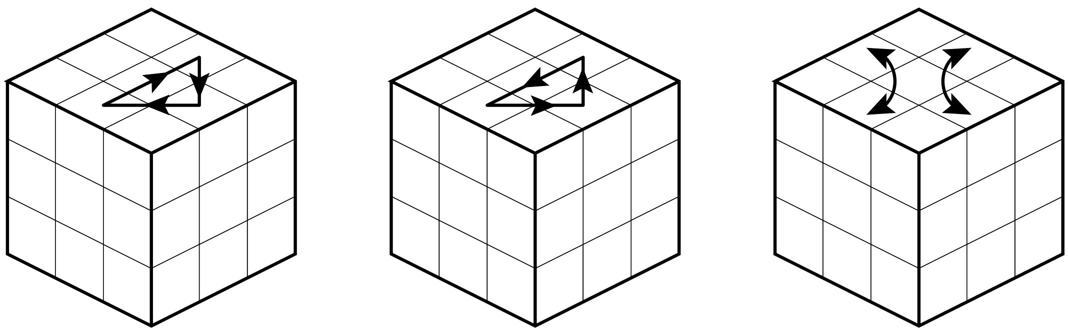
U'F'UF.URU'R'

If an edge of the middle slide in already on the middle slide but in the wrong place or twisted, then replace it with any of the edges of the top slide as we have just learned, and then apply the previous solution.

4 Solving the top slice

4.1 Putting edges in place

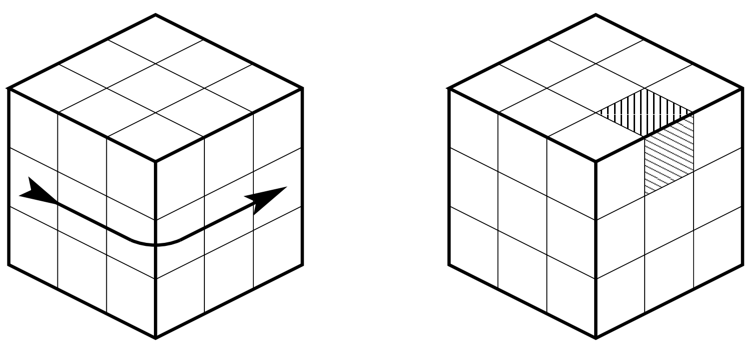
By turning the upper slide and looking at the four edges, there are three options: the four edges are in place (but maybe flipped), only one is in place, or no one is in place and there are two couples of edges swapped. In the last two cases, do the appropriate movement:



F.URU’R’.F’   F.RUR’U’.F’   F.URU’R’.F’.U^2.F.RUR’U’.F’

4.2 Curing flipped edges

An even number of edges are flipped. In order to put them right, we need a new movement M, a 90^o (i.e. a quarter turn) clockwise rotation of the middle slide, as shown in the left part of the next figure. In order to flip the hatchet edge, apply the move in the right part of the figure:



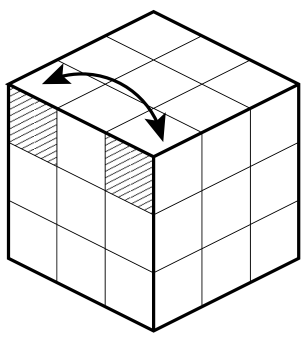
M   RM.RM.RM.RM

This movement DESTROYS the bottom and middle slides; but WITHOUT CHANGING THE CUBE ORIENTATION, turn the upper slide so that another edge that need to be flipped moves to the hatchet position, and APPLY AGAIN the move (RM)^4. The bottom and middle slides are now restored, and we have flipped two upper slide edges. If there still remain a couple of edges to be flipped, repeat the whole process one more time.

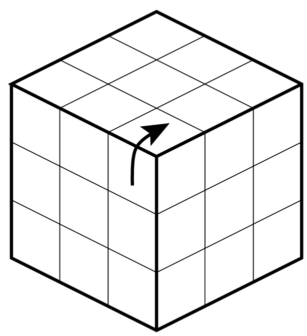
4.3 Putting corners in place

The first movement of the next figure swaps two corners of the upper slide, and DESTROYS the bottom slide; but by turning the upper slide WITHOUT CHANGING THE CUBE ORIENTATION, and applying this move again, the bottom slide is restored, and we have swapped another pair of upper slide corners. By repeating this move an EVEN number of times, the four corners of the upper slide can be put in place, but may be they are twisted.

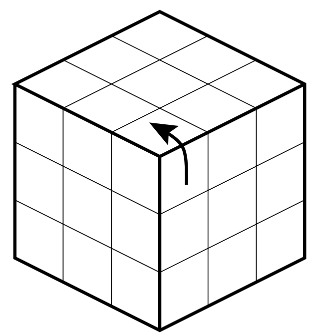
The number and the order of swaps to be made in order to put the corners in place must be CAREFULLY PLANNED in advance.



FRF’R’.FRF’R’.FRF’R’



FDF’D’.FDF’D’



DFD’F’.DFD’F’

4.4 Curing twisted corners

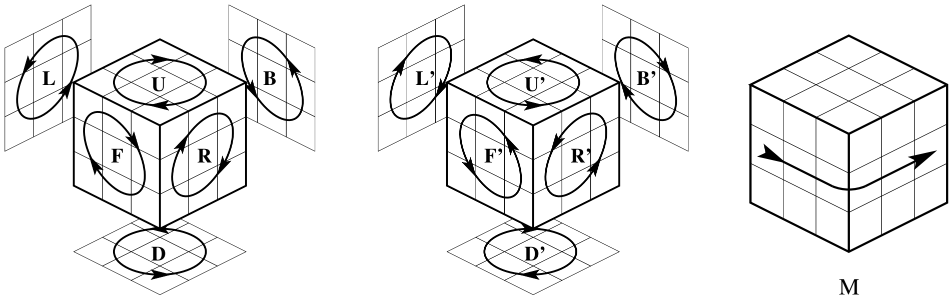
In order to twist the upper corners right, we apply the two last moves in the previous figure. Notice that these moves DESTROY the bottom and middle slides; but WITHOUT CHANGING THE CUBE ORIENTATION, and rotating the upper slide, we can successively twist all the upper slide corners. At the end of the process, the bottom and middle slides will be restored.

A last warning: the movements in the last three stages (4.2, 4.3, 4.4) are very tricky, because they destroy the middle and bottom slides and they are not restored until the end of the stage. Therefore you must be very careful and DO NOT CHANGE THE CUBE ORIENTATION until the end of the stage.

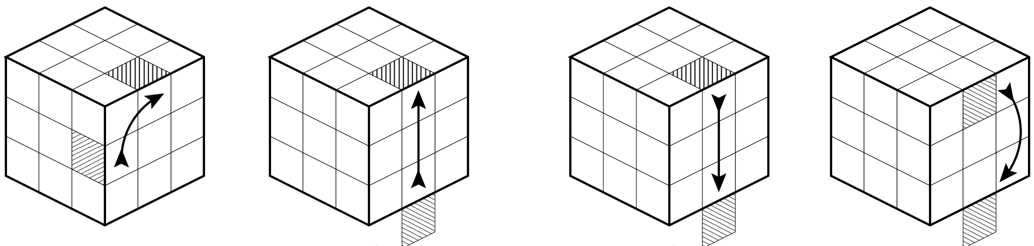
5 Final optional step

Shout HURRAAAAAAAAH!!!! and send some presents to Paco, or buy a round of drinks, or any other pleasant option.

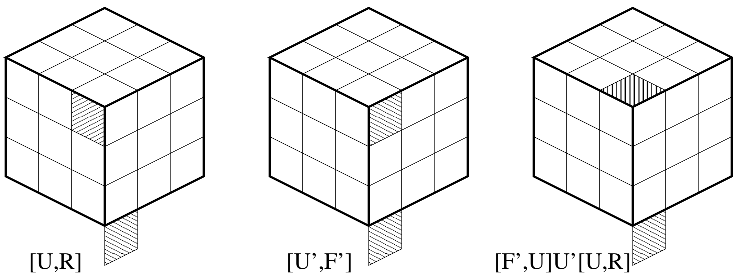
Rubik's cube solution summary



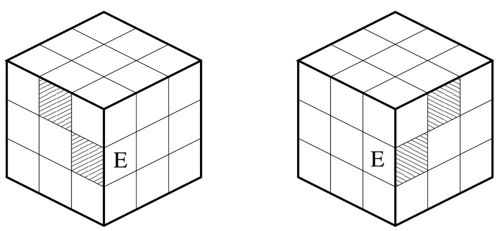
M



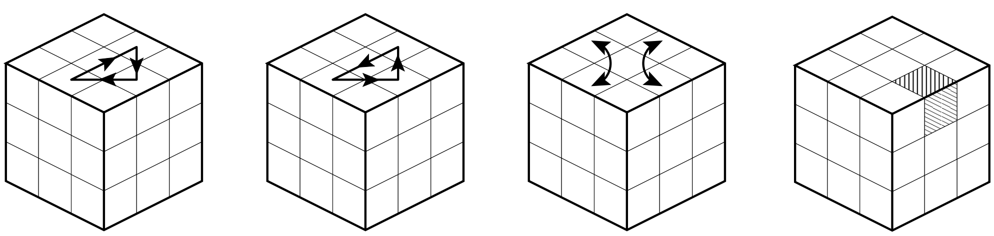
R (edges aloft) R^2 R^2 (in place) UFR'F'



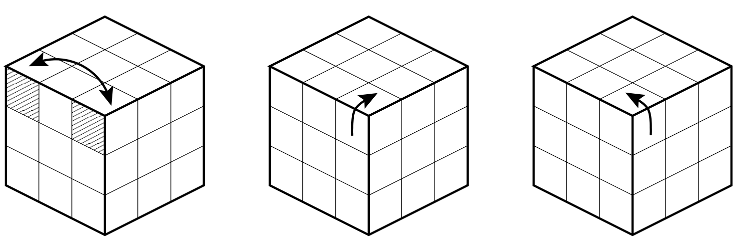
[U,R] [U',F'] [F',U]U'[U,R]



[U,R][U',F'] [U',F'][U,R]



F[U,R]F' F[R,U]F' F[U,R]F'U^2F[R,U]F' $(RM)^4$



$[FR]^3$ $[F,D]^2$ $[D,F]^2$

Extras

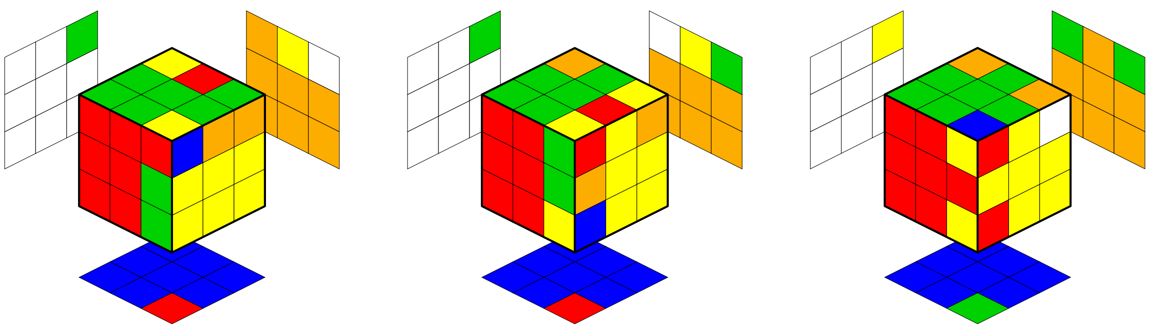
We are using the symbol ' for doing the movements in reverse order. We will write

$$(\mathrm{AC}\ldots\mathrm{EG})' = \mathrm{G'E'}\ldots\mathrm{B'A'}.$$

Using the symbol I for *doing nothing* (the *Identity*, as it is called in Mathematics), we have AA' = A'A = I. Playing with the definitions is easy to see that [A,C]' = [C,A].

Commutator properties

The next figure shows the effect of the commutator [U,R] and its powers on the Rubik's cube.



[U,R] $\quad$ $[\mathrm{U,R}]^2$ $\quad$ $[\mathrm{U,R}]^3$

[U,R] alters seven cubes, three edges and four corners. The two front corners are swapped, and the two back corners are also swapped. And the three edges are out of place.

$[\mathrm{U,R}]^2$. The four corners are all in place, but twisted, while the three edges are out of place. This movement leaves the bottom slide unaltered, except that it twists one corner. The whole stage (4.4) is based on this observation.

$[\mathrm{U,R}]^3$. The three edges are in place, and the corners are swapped in pairs: the two front corners and the two back corners are swapped. In particular, the front face is unmodified, except for the the swapping of two corners. The whole stage (4.3) is based on this observation.

$[\mathrm{U,R}]^6$ is the identity, it does nothing!. $[\mathrm{U,R}]^6 = \mathrm{I}$.

This small analysis should help to understand the moves solving the cube.

The only complex movement not based on the commutator is $(\mathrm{RM})^4$ in the stage (4.2). It could be replaced by $(\mathrm{F[U,R]F'U'})^2$, based on the commutator, that flips two edges on the top slide without modifying the middle and bottom slides, but $(\mathrm{RM})^4$ is simple and beautiful.