Comparison of Several Approaches to the Relativistic Dynamics of Directly Interacting Particles

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Several approaches to the relativistic dynamics of directly interacting particles are compared. The equivalence between constrained Hamiltonian relativistic systems and a priori Hamiltonian predictive ones is completely proved. Coordinate transformations are obtained to express these systems in the framework of noncovariant predictive mechanics. The world line condition for constrained Hamiltonian relativistic systems is analyzed and is proved to be also necessary in the predictive Hamiltonian framework.

1. Introduction

In the study of relativistic N particle systems with direct interaction, several methods have been proposed and throughly established by the corresponding streams of papers issued in the literature. The common aim of most of these approaches is to carry out the so-called Dirac program [1]; i.e., to construct a symplectic realization of the Poinvaré group $\mathfrak{P}$ on a $N \times d$ dimensional phase space Γ. The scope of the present work will be restricted to spinless particles, so that we shall have $d = 6$.

Among the different approaches appearing up to the present time we shall consider and compare here with some detail the following ones:

(a) Predictive relativistic mechanics in the manifestly predictive formalism [2] (PRM-3).

(b) The manifestly covariant formulation of the predictive relativistic mechanics [3] (PRM-4), restricting our study to the a priori Hamiltonian predictive systems [4] (HPS).

(c) The constrained Hamiltonian systems (CHS) as have been used by Rohrlich and others [5].

Another approach which must be mentioned here is the singular Lagrangian

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formalism [24]. However, we do not consider it in the present work because the equivalence between this approach and both formalisms of predictive relativistic mechanics and constrained Hamiltonian dynamics has been already established elsewhere [25].

In Section 2 we present the main ideas and the general features of each framework. After that, in Section 3, the CHS formalism is developed in detail using the language of differential geometry. A derivation of the world line condition is presented there making clear the two different actions of the Poincaré group on both the phase space of our dynamical system and the Minkowsky space where the world lines of particles occur.

In Section 4 the HPS formalism is analized in detail and a kind of world line condition is proved to be necessary in this framework as well. In Section 5, the equivalence of CHS and HPS formalisms is proved, meaning that the same mathematical objects can be introduced and worked either in one framework or the other, yielding, at the end, the same world lines for the individual particles.

In Section 6 (resp. 7) we show how a constrained relativistic Hamiltonian system (resp. a Hamiltonian predictive system) can be formulated in terms of noncovariant predictive relativistic mechanics.

Finally, we want to warn the reader against the danger of assigning the results presented here to a range of validity beyond the purely local level (i.e., some open neighborhood of each point in the respective domain). This limitation is due to the repeated use of the implicit function theorem all along the present paper. Although we have emphasized these limitations at some theorems, we have renounced to keep the mathematical rigour in this point (i.e., to formulate local statements for results which only hold locally) for simplicity and to not puzzler the reader with too many unessential details which would make the reading much harder and hide the intuitive geometrical ideas underlying the main stream of reasoning in the paper.

Mention should also be made here of a related previous work Lusanna [26], where a review of several approaches to relativistic dynamics with direct interaction is presented.

2. General Features of the Frameworks

2a. *PRM-3*

This approach starts from the following two premises:

(i) The equations of motion are "like Newton": The phase is $T\mathbb{R}^{3N}$ labeled by the positions x_a^i and velocities v_b^j of the particles and, in any inertial frame, the motion is described by a second-order differential system

$$\frac{dx_a^i}{dt} = v_a^i, \qquad \frac{dv_a^i}{dt} = \mu_a^i(x_b^j, v_c^l, t), \qquad a, b, c,\ldots = 1,\ldots, N; i, j, l,\ldots = 1, 2, 3. \quad (2.1)$$

We shall assume also that the accelerations μ_a^i depend on the masses of the particles $(m_1,\ldots, m_N)$.

The general solution of (2.1) will be written as $\psi_a^i(t, x_b^j, v_c^l)$, fulfilling the initial conditions

$$\psi_a^i(0, x_b^j, v_c^l) = x_a^i, \qquad \frac{\partial \psi_a^i}{\partial t}(0, x_b^j, v_c^l) = v_a^i. \tag{2.2}$$

(ii)   The equations of motion (2.1) hold in any inertial frame and the standard realization of $\mathfrak{P}$ on M_4 transforms the N world lines of the particles seen from an initial frame into the world lines observed from another.

This means that, if

$$\gamma_a \equiv \{x_a^v(t) \equiv (t, \psi_a^i(t, x_0, v_0)), t \in R\}, \qquad a = 1,..., N,$$

is the world line of particle (a) seen by a given inertial observer $\mathfrak{S}$ when the system starts from the initial state $(x_0, v_0) \in T\mathbb{R}^{3N}$, then for another observer $\mathfrak{S}'$ related to $\mathfrak{S}$ by the Poincaré transformation $(L_v^\mu, A^\mu) \in \mathfrak{P}$, another set of initial data $(x_0', v_0') \in T\mathbb{R}^{3N}$ must exist such that the world lines

$$\gamma_a' \equiv \{x_a'^\mu(t') = (t', \psi_a^i(t', x_0', v_0')), t' \in R\}, \qquad a = 1,..., N,$$

coincide with the former ones: $\gamma_a' = \gamma_a$, $a = 1,..., N$.

Differentiating in respect to the parameters of the Poincaré group in the neighbourhood of the identity, we obtain that the relativistic invariance of the world lines is equivalent to the three conditions

(i)   the acceleration functions μ_a^i do not depend on t explicitly,

(ii)   they are also invariant under space translations and behave like spacevectors under rotations, and

(iii)   satisfy the so called Currie–Hill equations [2]:

$$\sum_{b=1}^{N} \left\{ v_b^j(x_b^k - x_a^k)\frac{\partial \mu_a^i}{\partial x_b^i} + \{\mu_b^j(x_b^k - x_a^k) + v_b^j v_a^k - \delta^{jk}\}\frac{\partial \mu_a^i}{\partial v_b^j} \right\}$$
$$= 2\mu_a^i v_a^k + v_a^i \mu_a^k. \tag{2.3}$$

Furthermore, conditions (ii) and (iii) are satisfied if, and only if, the vector fields

$$\mathbf{P}_0 = \sum_{a=1}^{N} \left\{ v_a^i \frac{\partial}{\partial x_a^i} + \mu_a^i(x, v)\frac{\partial}{\partial v_a^i} \right\}, \tag{2.4a}$$

$$\mathbf{P}_i = -\sum_{a=1}^{N} \frac{\partial}{\partial x_a^i}, \tag{2.4b}$$

$$\mathbf{J}_l = \sum_{a=1}^{N} \varepsilon_l{}^i{}_k \left\{ x_a^k \frac{\partial}{\partial x_a^i} + v_a^k \frac{\partial}{\partial v_a^i} \right\}, \tag{2.4c}$$

$$\mathbf{K}_j = \sum_{a=1}^{N} \left\{ x_{aj} v_a^i \frac{\partial}{\partial x_a^i} + (v_{aj} v_a^i + x_{aj}\mu_a^i - \delta_j^i)\frac{\partial}{\partial v_a^i} \right\} \tag{2.4d}$$

generate a realization of $\mathfrak{P}$ on $T\mathbb{R}^{3N}$.

Up to this point, one step in the Dirac program has been attained but we have still to define a symplectic form on $T\mathbb{R}^{3N}$, i.e., a nonsingular Poisson bracket $\omega \in \Lambda^2(T\mathbb{R}^{3N})$, invariant under the realization of $\mathfrak{P}$. This is equivalent to

$$\mathfrak{L}(\Lambda_I)\omega = 0, \qquad I = 1,..., 10, \tag{2.5}$$

where $\mathfrak{L}$ is the Lie derivative and Λ_I means anyone of the 10 vector fields (2.4).

Thanks to the Poincaré lemma [6] and to the fact that ω will be a closed form ($d\omega = 0$), Eqs. (2.5) will guarantee the existence, at least locally, of 10 generating functions Λ_I, $I = 1,..., 10$, such that

$$d\Lambda_I = -i(\Lambda_I)\omega \qquad \text{or} \qquad \Lambda_I = \{\Lambda_I, \ \},$$

where $i(\ )$ is the interior product and $\{\ ,\ \}$, the Poisson bracket associated to ω.

Moreover, in the spaial case of the Poincaré group we shall be able to choose [7] the generating functions Λ_I in such a way that

$$\{\Lambda_I, \Lambda_J\} = C_{IJ}^K \Lambda_K, \qquad I, J, K = 1,..., 10,$$

where C_{IJ}^K are the structure constants of $\mathfrak{P}$ in the corresponding parametrization

$$[\Lambda_I, \Lambda_J] = C_{IJ}^K \Lambda_K.$$

At this point the so called no-interaction theorems [8] state that, unless the accelerations μ_a^i vanish, there is no simplectic form ω such that:

  (i)   it is invariant under the ten vector fields (2.4) and

  (ii)   the coordinates x_a^i are canonical ones referred to ω.

Only free-particle systems can therefore be fitted in this framework if both conditions (i) and (ii) are required. This obstruction is commonly circumvented by dispensing condition (ii). Then the problem is that too many symplectic forms are permitted by only the invariance condition (i); however, each physical problem suggests a suitable set of boundary conditions which selects a smaller family of solutions and, in some specially interesting cases, a unique result is obtained [9].

Summarizing, a PRM-3 is given by a symplectic realizations of $\mathfrak{P}$ on the symplectic manifold $(T\mathbb{R}^{3N}, \omega)$ such that in a given coordinate frame (x, v) the infinitesimal generators of $\mathfrak{P}$ have the simple shape (2.4).

2b. PRM-4

In this approach the phase space is TM_4^N labelled by (x_a^μ, π_b^ν), where the Poincaré group acts in the standard way. Hence the infinitesimal generators are

$$\mathbf{P}_\mu = -\sum_{a=1}^N \frac{\partial}{\partial x_a^\mu},$$

$$\mathbf{J}_{\mu\nu} = \sum_{a=1}^N \left\{ x_{a\mu} \frac{\partial}{\partial x_a^\nu} - x_{a\nu} \frac{\partial}{\partial x_a^\mu} + \pi_{a\mu} \frac{\partial}{\partial \pi_a^\nu} - \pi_{a\nu} \frac{\partial}{\partial \pi_a^\mu} \right\}, \tag{2.6}$$

where greek indices run from 0 to 3 and are raised and lowered by the Minkowsky metrics $\eta_{\mu\nu} = (-+++)$.

It is then assumed as the fundamental hypothesis that for any initial state of the system, there exists one world line in M_4 for each particle.

Starting from a given set of initial data $z = (x_a^\mu, \pi_b^\nu) \in TM_4^N$, the world line of (a) is given by $\varphi_a^\mu(\tau_a, z)$, where the parameter τ_a is the proper time along the world line of the particle (a) divided by its mass m_a and the initial condition $\varphi_a^\mu(0, z) = x_a^\mu$ is assumed. Furthermore, the four-velocity of (a) evolves according to $(\partial \varphi_a^\mu / \partial \tau_a)(\tau_a, z)$ with the initial condition

$$\frac{\partial \varphi_a^\mu}{\partial \tau_a}(0, z) = \pi_a^\mu.$$

Also, since τ_a is an affine parameter and thanks to the initial condition

$$\left\{ \frac{\partial \varphi_a^\mu}{\partial \tau_a} \cdot \frac{\partial \varphi_{a\mu}}{\partial \tau_a} \right\}(\tau_a, z) = \pi_a^\mu \pi_{a\mu} = -m_a^2, \qquad (2.7)$$

Therefore, for each $z \in TM_4^N$ we have a world submanifold

$$S(z) \equiv \{ \Phi(\tau_1, ..., \tau_N; z), (\tau_1, ..., \tau_N) \in \mathbb{R}^N \},$$

where

$$\Phi(\tau_1, ..., \tau_N; z) \equiv \left(\varphi_a^\mu(\tau_a, z), \frac{\partial \varphi_b^\nu}{\partial \tau_b}(\tau_b, z) \right) \in TM_4^N, \qquad (2.8)$$

That is to say, $\Phi(\tau_1, ..., \tau_N; z)$ is the state of the system after having moved each particle (a) an amount τ_a along its own world line.

The assumed existence of well-behaved world lines implies that, if we take $\Phi(\tau_1, ...; z)$ as new initial data, we must obtain the same world manifold

$$S(z) = S(\Phi(\tau_1, ..., \tau_N; z))$$

with the labeling shifted according to $(\tau_1, ..., \tau_N)$; that is,

$$\Phi\{\tau_1', ..., \tau_N'; \Phi(\tau_1, ..., \tau_N; z)\} = \Phi(\tau_1 + \tau_1', ..., \tau_N + \tau_N'; z). \qquad (2.9)$$

This is equivalent to require the map $\Phi: \mathbb{R}^N \times TM_4^N \to TM_4^N$ to define a realization of the abelian group $\mathbb{R}^N$ on TM_4^N. Furthermore, due to the well-known properties of the action of a Lie group on a manifold, the latter result occurs if, and only if, the infinitesimal generators $\mathbf{H}_a$ commute

$$[\mathbf{H}_a, \mathbf{H}_b] = 0, \qquad a, b = 1, ..., N. \qquad (2.10)$$

Moreover, due to definition (2.8) these generators act on the coordinate functions (x_a^μ, π_b^ν) as

$$\mathbf{H}_a x_b^\mu = \delta_{ab} \cdot \pi_a^\mu, \qquad \mathbf{H}_a \pi_b^\mu = \delta_{ab} \cdot \theta_a^\mu, \qquad (2.11)$$

where the functions $\theta_a(x, \pi)$ account for the four-accelerations of the particles and, due to the affine character of τ_a, satisfy

$$\theta_a^\mu(x, \pi) \cdot \pi_{a\mu} = 0. \tag{2.12}$$

The relativistic invariance obviously requires the accelerations θ_a^μ to behave as a translation invariant four-vector or, equivalently,

$$[\mathbf{H}_a, \Lambda_I] = 0, \qquad a = 1,..., N; I = 1,..., 10, \tag{2.13}$$

where Λ_I represents any one of the generators of $\mathfrak{P}$.

From the commutation relations (2.10) and (2.13) we have that the $10 + N$ vector fields $\mathbf{H}_a$, $a = 1,..., N$; Λ_I, $I = 1,..., 10$ generate a realization of the Lie group $\mathfrak{G}_N = \mathbb{R}^N \otimes \mathfrak{P}$ on TM_4^N. This is called the full symmetry group of the N-particle system. The action on TM_4^N of a given $(\tau_a; \varepsilon_I) \in \mathfrak{G}_N$ will be written as

$$G(\tau_a; \varepsilon_I) = \Phi(\tau_1,..., \tau_N) \circ g(\varepsilon_I) = g(\varepsilon_I) \circ \Phi(\tau_1,..., \tau_N),$$

where $g(\varepsilon_I)$ is the standard action of $\mathfrak{P}$ on TM_4^N and $\Phi(\tau_1,..., \tau_N)$, which has been defined in (2.8), will be expressed in the following as

$$\Phi(\tau_1,..., \tau_N) \equiv \exp\left(\sum_{a=1}^{N} \tau_a \mathbf{H}_a\right).$$

Condition (2.12) guarantees that the mass shell

$$\mathfrak{M}(m_1,..., m_N) = \{(x_a, \pi_b) \in TM_4^N / \pi_b^\mu \pi_{b\mu} = -m_b^2, b = 1,..., N\}$$

is left invariant by this realization of $\mathfrak{G}_N$.

Up to this point we have that a PRM-4 of an N-particle system is given by a realization of $\mathfrak{G}_N$ on TM_4^N such that:

(i) The N infinitesimal generators $\mathbf{H}_a$ have the simple form

$$\mathbf{H}_a = \pi_a^\mu \frac{\partial}{\partial x_a^\mu} + \theta_a^\mu(x, \pi) \frac{\partial}{\partial \pi_a^\mu}, \tag{2.14}$$

where the four-accelerations θ_a^μ satisfy the orthogonality condition (2.12) and

(ii) the subgroup $\mathfrak{P} \subset \mathfrak{G}_N$ acts on TM_4^N in the standard way.

If we now look for a symplectic form $\Omega \in \Lambda^2(TM_4^N)$ invariant under $\mathfrak{G}_N$, we find a covariant version of the no-interaction theorem [10]. Unless the accelerations $\theta_a^\mu(x, \pi)$ vanish, there is no symplectic from Ω fulfilling the two requirements of:

(i) being $\mathfrak{G}_N$ invariant and

(ii) the coordinates x_a^μ being canonical ones.

As in the case commented in Subsection (2a), condition (ii) is given up to

circunbent the no-interaction theorem and again too many symplectic forms $\mathfrak{G}_N$-invariant are permited.

At this point, the approach which we have called PRM-4 splits in two different directions:

(2b.1). The stream initiated by Bel and Martin [11], which considers that, for most of the classical relativistic interactions which have been already formulated [12] in terms of PRM-4, the trivial symplectic form of free particle systems appears as a natural boundary condition when the distances between particles approach infinity.

This fashion has the advantage that one is always sure of dealing with something which has to do with one of the classical relativistic interactions. However, it leads unfaillingly to intricated power expansions on some coupling constant.

In the scope of this paper this approach will not be considered.

(2b.2). The method proposed by Droz–Vincent [13], which assumes an a priori canonical realization of the full symmetry group on $T^*M_4^N$, and then a PRM-4 is obtained by finding out a suitable mapping into TM_4^N.

2.c. CHS

Here we shall only consider this formalism as it has been presented in Ref. [5]. This approach starts from $T^*M_4^N$ endowed with the canonical symplectic form

$$\Omega = \sum_{a=1}^{N} dp_\mu^a \wedge dq_a^\mu; \tag{2.15}$$

i.e., the Poisson brackets are

$$\{q_a^\mu, p_\nu^b\} = \delta_a^b \, \delta_\nu^\mu, \qquad \{q_a^\mu, q_b^\nu\} = \{p_\mu^a, p_\nu^b\} = 0,$$

$$a, b = 1,..., N, \qquad \mu, \nu = 0, 1, 2, 3,$$

and the standard action of $\mathfrak{P}$ on $T^*M_4^N$ is defined by the generating functions

$$P_\mu = \sum_{a=1}^{N} p_\mu^a, \qquad J_{\mu\nu} = \sum_{a=1}^{N} (q_{a\mu} p_\nu^a - q_{a\nu} p_\mu^a) \tag{2.16}$$

The mass shells of the particles are then defined by N Poincaré invariant functions $\{K_a \; a = 1,..., N\}$ on $T^*M_4^N$. We shall assume besides that these functions contain the squared masses of the particles as parameters in such a way that $\det(\partial K_a/\partial m_b^2) \neq 0$. So that when the masses are $m_1,..., m_N$ the mass shell is

$$\mathfrak{M}(m_1,..., m_N) = \{z = (q, p) \in T^*M_4^N/K_a(z, m_b) = 0\}. \tag{2.17}$$

Furthermore, the functions $K_a(z, m_b)$ are required to be first class on the mass shell $\mathfrak{M}(m_1,..., m_N)$ for any fixed value of the mass parameters

$$\{K_a(z, m_b), K_c(z, m_e)\} = 0 \qquad \text{on} \quad \mathfrak{M}(m_c), \tag{2.18}$$

and, since they are Poincaré invariant, we also have

$$\{K_a(z, m_b), \Lambda_I\} = 0, \tag{2.18'}$$

where Λ_I is any one of the generating functions (2.16).

By means of the Poisson bracket, the functions $K_a(z, m_b)$ define N vector fields $\mathbf{K}_a(m_b) \equiv \{K_a(z, m_b), \}$ which, due to Eq. (2.18), are tangent to the mass shell $\mathfrak{M}(m_c)$ and commute with each other:

$$|\mathbf{K}_a(m_c), \mathbf{K}_b(m_c)| = 0 \qquad \text{on } \mathfrak{M}(m_c).$$

They therefore generate a realization of $\mathbb{R}^N$ on $\mathfrak{M}(m_c)$. The orbits $\Sigma(z)$ of the points $z \in \mathfrak{M}(m_c)$ under the action of the group provide a foliation of the mass shell defining the restricted phase space

$$\Psi(m_c) \equiv \mathfrak{M}(m_c)/\Sigma,$$

the dimension of which is $6N$.

The latter can be represented by choosing a $6N$-submanifold $\Gamma_0(m_c) \subset \mathfrak{M}(m_c)$ such that $\forall z_1, z_2 \in \Gamma_0(m_c),\ z_1 \neq z_2 \Rightarrow \Sigma(z_1) \cap \Sigma(z_2) = \varnothing$.

The submanifold $\Gamma_0(m_c)$ is defined by introducing N "fixations" $\chi_a(z, m_c)$, $a = 1,..., N$, on $T^*M_4^N$ and the latter condition is guaranteed, at least locally, if and only if

$$\det(\{K_a, \chi_b\}) \neq 0. \tag{2.19}$$

However, this framework does not seem to be suitable enough for the "time evolution" to be represented. Hence, the "fixations" are relaxed by allowing them to depend on a parameter τ which will play the role of a "time." This one-parameter family of fixations defines the set of $6N$-submanifolds:

$$\Gamma_\tau(m_c) \equiv \{z \in T^*M_4^N / K_a(z, m_c) = 0, \chi_b(z, m_c, \tau) = 0, a, b, c = 1,..., N\},$$

each one being a "good" representative of $\Psi(m_c)$, at least locally, if and only if condition (2.19) holds for any value of τ.

The extended phase space is the $(6N + 1)$-submanifold

$$\Gamma'(m_c) \equiv \bigcup_{\tau \in \mathbb{R}} \Gamma_\tau(m_c),$$

which is locally defined by the N mass shell constraints K_a and by the $(N - 1)$ fixations which result after eliminating τ from $\chi_a(z, m_b, \tau) = 0$, $a = 1,..., N$.

Furthermore, as far as condition (2.19) is satisfied, there is a unique linear combination $\mathbf{H}$ of the vector fields $\mathbf{K}_a(m_c)$ such that it is tangent to $\Gamma'(m_c)$ and satisfies

$$\frac{\partial \chi_a}{\partial \tau} + \mathbf{H}\chi_a = 0 \qquad \text{on } \Gamma'(m_c). \tag{2.20}$$

This vector field is defined by

$$\mathbf{H} = -\sum_{b,c=1}^{N} \frac{\partial \chi_b}{\partial \tau} \cdot S^{bc} \cdot \mathbf{K}_c(m_a), \tag{2.21}$$

where S^{bc} is the inverse matrix of $D_{ab} = \{K_a, \chi_b\}$, i.e.,

$$\sum_{c=1}^{N} S^{bc} \cdot D_{ca} = \delta_a^b = \sum_{c=1}^{N} D_{ac} S^{cb}. \tag{2.22}$$

The vector field $\mathbf{H}$ generates the realization of the additive Lie group $\mathbb{R}$ on $\Gamma'(m_c)$ defined by the exponential map $\exp(\lambda \mathbf{H})$, $\lambda \in \mathbb{R}$ and it can be easily shown that $\exp(\lambda \mathbf{H}): \Gamma_\tau(m_c) \to \Gamma_{\tau+\lambda}(m_c)$.

It is therefore said $\mathbf{H}$ generates the evolution associated to the time τ.

Finally, the coordinates $q_a^\mu(z)$ are, by definition, identified with the position coordinates x_a^μ of the particle (a) when the state of the system is $z \in \Gamma'(m_c)$. Thus, when the system undergoes its time evolution starting from $z \in \Gamma'(m_c)$ the world line of (a) is

$$x_a^\mu(\tau) = q_a^\mu (\exp(\tau \mathbf{H})z).$$

The relativistic invariance of the system requires these world lines to be Poincaré invariant and this implies new conditions on the fixations χ_a, which are commonly called world line condition (WLC) |14|. (We shall come back to this point in the next section).

3. The Constrained Hamiltonian Systems

3a. *Equivalent Formulations of a Given CHS*

As has been seen in Section 2c, a CHS is characterized by

(CHS-i)   The mass shells $\mathfrak{M}(m_1,..., m_N)$ defined by N functions $K_a(z, m_c)$ containing the masses (m_c) as parameters,

(CHS-ii)   which are first class and Poincaré invariant on $\mathfrak{M}(m_c)$—see Eqs. (2.18) and (2.18'),

(CHS-iii)   the $(6N + 1)$-submanifold

$$\Gamma'(m_c) = \{z \in T^* M_4^N / K_a(z, m_c) = 0, \chi_b(z, m_c, \tau) = 0, \text{ for some } \tau \in \mathbb{R}\},$$

where the fixations fulfill condition (2.19), and

(CHS-iv)   the vector field $\mathbf{H}$ given by Eq. (2.21), which is defined on $\Gamma'(m_c)$, and tangent to it.

The following theorem states what changes can be done in the constraints preserving these features (CHS-(i) to (iv)).

T.3.1. THEOREM. *Let us consider a* CHS *defined by the constraints* $(K_1,...,K_N; \chi_1,...,\chi_N)$. *If we change them into*

$$\tilde{K}_a \equiv h_a(K_1,...,K_N); \qquad \tilde{\chi}_b \equiv f_b(K_1,...,K_N; \chi_1,...,\chi_N), \qquad (3.1)$$

where $(h_a, f_b)_{a,b=1,...,N}$ are $2N$-independent functions such that

$$h_a(0,...,0) = 0 \qquad and \qquad f_b(0,...,0) = 0, \qquad (3.2)$$

then the CHS $(\tilde{K}_1,...,\tilde{K}_N; \tilde{\chi}_1,...,\tilde{\chi}_N)$ *is equivalent to the former one, i.e., transformation* (3.1) *preserves* CHS-(i) *to* (iv).

Proof. (i) and (iii) are obviously preserved since Eqs. (2.1) and (2.2) guarantee:

$$\tilde{\mathfrak{M}}(m_c) = \mathfrak{M}(m_c) \qquad and \qquad \tilde{\Gamma}'(m_c) = \Gamma'(m_c)$$

(ii)

$$\{\tilde{K}_a, \tilde{K}_b\} = \sum_{c,d=1}^{N} \frac{\partial h_a}{\partial K_c} \cdot \frac{\partial h_b}{\partial K_d} \cdot \{K_c, K_d\} = 0 \qquad on \quad \mathfrak{M}(m_c) = \tilde{\mathfrak{M}}(m_c).$$

(iv) Let us consider the matrix

$$\tilde{D}_{ab} = \{\tilde{K}_a, \tilde{\chi}_b\} = \sum_{c,d=1}^{N} \frac{\partial h_a}{\partial K_c} \cdot \frac{\partial f_b}{\partial \chi_d} \cdot D_{cd}. \qquad (3.3)$$

Since $(h_a, f_b)_{a,b=1,...,N}$ is a set of $2N$ independent functions and h_a does not depend on χ_c, neither $(\partial f_b/\partial \chi_c)_{b,c=1,...,N}$ nor $(\partial h_a/\partial K_d)_{a,d=1,...,N}$ can be singular. Then, since $\tilde{D}_{ab}$ is a product of three nonsingular matrices, it is itself nondegenerate.

From Eq. (3.3) we have that the inverse $\tilde{S}^{ab}$ of $\tilde{D}_{bc}$ is related to S^{ab} by

$$S^{cd} = \sum_{a,b=1}^{N} \frac{\partial f_b}{\partial \chi_c} \cdot \frac{\partial h_a}{\partial K_d} \cdot \tilde{S}^{ba},$$

and, consequently, we have for the vector field H that

$$\tilde{\mathbf{H}} = - \sum_{a,b=1}^{N} \frac{\partial \tilde{\chi}_b}{\partial \tau} \cdot \tilde{S}^{ba} \cdot \{\tilde{K}_a, \} = - \sum_{c,d=1}^{N} \frac{\partial \chi_c}{\partial \tau} \cdot S^{cd} \cdot \{K_d, \} = \mathbf{H} \qquad (3.4)$$

and the proof is over.

As a consequence of this result, a given CHS parametrized by the masses can always be formulated in the following terms:

(a) First, we find out the half square masses from the constraints $K_a(z, m_c)$, $a = 1,...,N$. This can be always be done, since we have assumed that

$\det(\partial K_a/\partial m_b^2) \neq 0$ and it is equivalent to combine them until obtaining the new constraints

$$\tilde{K}_a(z, m_b) = H_a(z) + \tfrac{1}{2}m_a^2. \tag{3.5}$$

Then the mass shell $\mathfrak{M}(m_c)$ is defined by the functions $H_a(z)$ (which do not depend on the masses anymore) according to

$$\mathfrak{M}(m_c) = \{z \in T^*M_4^N / H_a(z) + \tfrac{1}{2}m_a^2 = 0, a = 1, ..., N\}. \tag{3.6}$$

Furthermore, the vector field $\tilde{K}_a$ defined on and tangent to $\mathfrak{M}(m_c)$ are given by

$$\mathbf{\tilde{K}}_a \equiv \{\tilde{K}_a(z, m_c),\ \} = \{H_a(z),\ \} \qquad \text{on } \mathfrak{M}(m_c), \tag{3.7}$$

which have the advantage of being independent of the masses and can be considered as the restriction to $\mathfrak{M}(m_c)$ of N other more general vector fields $\mathbf{H}_a = \{H_a,\ \}$ defined on the whole $T^*M_4^N$.

(b)   Second, we combine the constraints $K_a(z, m_c)$ and the fixations obtaining a new set of $\tilde{\chi}(z, m_c, \tau)$ that:

(i)   do not depend explicitly on the masses,

(ii)   the latter $(N - 1)$ fixations do not depend on τ:

$$\tilde{\chi}_A(z), \qquad A = 2, ..., N, \tag{3.8a}$$

(iii)   the parameter τ apears only in the first fixation:

$$\tilde{\chi}_1(z, \tau) = h(z) - \tau. \tag{3.8b}$$

This is possible because at least one of the former fixations χ_a, $a = 1, ..., N$ must depend on τ explicitly.

What we have reached up to this point is that any CHS parametrized by the masses can be always formulated in terms of

$$\tilde{K}_a(z, m_b) = H_a(z) + \tfrac{1}{2}m_a^2, \qquad a = 1, ..., N, z \in T^*M_4^N,$$
$$\tilde{\chi}_A(z, m_b) = \tilde{\chi}_A(z), \qquad A = 2, ..., N, \tag{3.9}$$
$$\tilde{\chi}_1(z, m_b, \tau) = h(z) - \tau$$

(for the sake of simplicity the tildes will be omitted hereafter).

3b. *The Action of the Poincaré Group on Γ_τ*

Most of the results presented in this subsection and in the next one have been already obtained by Sudarshan *et al.* [14]. However, we are goind to give a slightly different presentation which, together with the one of Ref. [14], will provide a deeper insight into the way this formalism works.

Once the CHS is formulated in the simplest form (3.9) we realize that the vector fields

$$\mathbf{H}_a = \{H_a, \ \}, \qquad \mathbf{P}_\mu = \{P_\mu, \ \}, \qquad \mathbf{J}_{\mu\nu} = \{J_{\mu\nu}, \ \}$$

satisfy the commutation relations (2.10) and (2.13). Hence, they generate a realization of $\mathfrak{G}_N$ on $T^*M_4^N$:

$$G(\tau_a, \varepsilon_I)\colon T^*M_4^N \to T^*M_4^N$$

$$z \to \varepsilon(\tau_a, \varepsilon_I)z = \exp\left(\sum_{a=1}^N \tau_a \mathbf{H}_a\right) \circ g(\varepsilon_I)z, \qquad (3.10)$$

where $\tau_a \in \mathbb{R}$, $a = 1,..., N$, $(\varepsilon_I) = (\varepsilon_1,..., \varepsilon_{10}) \in \mathfrak{P}$ and $g(\varepsilon_I)$ is the standard action of $\mathfrak{P}$ on $T^*M_4^N$.

Obviously, this action of $\mathfrak{G}_N$ does not leave $\Gamma'(m_c)$ invariant, that is, for a given $(\tau_a, \varepsilon_I) \in \mathfrak{G}_N$, $G(\tau_a, \varepsilon_I)\Gamma'(m_c) \neq \Gamma'(m_c)$.

However, the following two theorems state that this realization of $\mathfrak{G}_N$ induces a realization of $\mathfrak{P}$ on each slice $\Gamma_\lambda(m_c) \subset \Gamma'(m_c)$ defined by the constraints (3.9) for any chosen value of λ.

T.3.2. THEOREM. *Given* $z \in \Gamma_\lambda$ *and* $(\varepsilon_I) \in \mathfrak{P}$, *a unique N-tuple* $(\tau_1(z, \varepsilon_I),..., \tau_N(z, \varepsilon_I)) \in \mathbb{R}^N$ *exists such that*

$$G(\tau_a(z, \varepsilon_I), \varepsilon_J)\, z \in \Gamma_\lambda. \qquad (3.11)$$

(Actually, this assertion holds in an open neighbourhood of the identity rather than on the whole Poincaré group (see Fig. 1).)

Proof. The numbers $\tau_a(z, \varepsilon_I)$, $a = 1,..., N$, which we are looking for, are the roots of the N-equation system

$$\chi_a(G(\tau_b, \varepsilon_I)z, \lambda) = 0, \qquad a = 1,..., N, \qquad (3.12)$$

where, since the particular formulation (3.9) has been adopted, the parameter λ is given by $\lambda = h(z)$.

A unique N-tuple solution will exist for system (3.12) as far as the implicit function theorem (IFT) can be applied. And this is the case since, due to the condition (2.19), the matrix

$$\left(\frac{\partial \chi_a}{\partial \tau_b}\right)_{(G(\tau_a, \varepsilon_I)z, h(z))} = \{K_b, \chi_a\}_{(G(\tau_a, \varepsilon_I)z, h(z))} \qquad (3.13)$$

is nondegenerate on $\Gamma'(m_c)$, i.e., $\tau_a = 0$, $\varepsilon_I = 0$.

Also, it obviously follows that $\tau_a(0,..., 0; z) = 0$, $a = 1,..., N$.
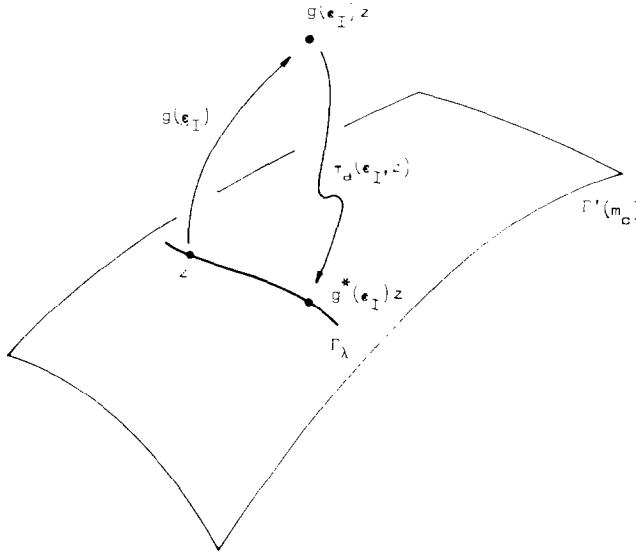
FIG. 1.   The Poincaré transformation $g(\varepsilon_I)$ takes z out from $\Gamma'(m_c)$ and $\exp(\sum_{h=1}^{N} \tau_h(\varepsilon_I, z)\mathbf{H}_h)$ brings it back to $\Gamma_\lambda(m_c)$.

This result allows us to define the transformation

$$
g^*(\varepsilon_I)\colon \ \Gamma'(m_c) \to \Gamma'(m_c),
$$
$$
z \to g^*(\varepsilon_I)z \equiv G(\tau_a(\varepsilon_I, z), \varepsilon_J)z
$$
(3.14)

for any $(\varepsilon_I) \in \mathfrak{P}$ (or, at least, in a neighbourhood of $(0) \in \mathfrak{P}$).

T.3.3. THEOREM. *Transformation* (3.14) *define a realization of* $\mathfrak{P}$ *on* $\Gamma'(m_c)$ *leaving each* $\Gamma_\lambda(m_c)$ *invariant* $(\forall \lambda \in \mathbb{R})$.

Proof.   The invariance of $\Gamma_\lambda(m_c)$ follows in an obvious way from the construction of $\tau_a(\varepsilon_I, z)$ in (T.3.2.).

Let us now consider two elements (ε_I^1), $(\varepsilon_J^2) \in \mathfrak{P}$ and let (ε_K) be its product: $(\varepsilon_K) = (\varepsilon_I^1) \circ (\varepsilon_J^2)$.

According to definitions (3.10) and (3.14) for a given $z \in \Gamma_\lambda \subset \Gamma'(m_c)$, we shall have

$$
g^*(\varepsilon_I^1) \circ g^*(\varepsilon_J^2)z \in \Gamma_\lambda(m_c), \qquad \lambda = h(z)
$$

and

$$
g^*(\varepsilon_I^1) \circ g^*(\varepsilon_J^2)z = \exp\left\{\sum_b \tau_b(\varepsilon_I^1, \tilde{z}) \cdot \mathbf{H}_b\right\} \circ g(\varepsilon_I^1)
$$
$$
\circ \exp\left\{\sum_a \tau_a(\varepsilon_J^2, z) \cdot \mathbf{H}_a\right\} \circ g(\varepsilon_J^2)z, \qquad (3.15)
$$

where $\tilde{z} \equiv g^*(\varepsilon_J^2)z$.

Then, thanks to the commutation relations (2.13), we can commute the exponential maps and the Poincaré transformation $g(\varepsilon_I^1)$ appearing in the right-hand side of Eq. (3.15), obtaining

$$g^*(\varepsilon_I^1) \circ g^*(\varepsilon_J^2) = \exp\left\{\sum_a [\tau_a(\varepsilon_I^1, \tilde{z}) + \tau_a(\varepsilon_J^2, z)] \cdot \mathbf{H}_a\right\} \circ g(\varepsilon_K)z \in \Gamma_{.1}(m_c).$$

The uniqueness theorem (T.3.2.) then implies that

$$\tau_a(\varepsilon_I^1, \tilde{z}) + \tau_a(\varepsilon_J^2, z) = \tau_a(\varepsilon_K, z), \tag{3.16}$$

where $\tilde{z} = g^*(\varepsilon_J^2)z$ and $(\varepsilon_K) = (\varepsilon_I^1) \circ (\varepsilon_J^2) \in \mathfrak{P}$.

Finally, introducing (3.16) into (3.15) and taking (3.14) into account we have

$$g^*(\varepsilon_I^1) \circ g^*(\varepsilon_J^2)z = g^*(\varepsilon_K)z, \qquad \forall z \in \Gamma'(m_c)$$

and the theorem is proved.

If we now look for the infinitesimal generators Λ_I^* for this realization of $\mathfrak{P}$ on $\Gamma'(m_c)$, we have that

$$\Lambda_I^* f(z) \equiv \left\{\frac{\partial}{\partial \varepsilon_I} f(g^*(\varepsilon_I)z)\right\}_{\varepsilon_J = 0}$$

$$= \Lambda_I f(z) + \sum_{a=1}^N \frac{\partial \tau_a}{\partial \varepsilon_I}(0, z) \cdot \mathbf{H}_a f(z) \tag{3.17}$$

for any function f on $\Gamma'(m_c)$ and any $z \in \Gamma'(m_c)$.

That is, the generator Λ_I^* on $\Gamma'(m_c)$ can be obtained by adding a suitable linear combination of $\mathbf{H}_a$, $a = 1,..., N$, to the corresponding Λ_I.

The following result, which was already pointed out by Sudarshan *et al.* [14], permits one to obtain Λ_I^* without working out the partial derivatives $\partial \tau_a / \partial \varepsilon_I$.

P.3.4. PROPOSITION. *For each infinitesimal generator of* $\mathfrak{P}$ $(\Lambda_I, I = 1,..., 10)$ *a unique linear combination* $\sum_{a=1}^N b_I^a \mathbf{H}_a$ *exists such that* $\Lambda_I + \sum_{a=1}^N b_I^a \mathbf{H}_a$ *is tangent to* $\Gamma'(m_c)$.

The proof is very easy, the uniqueness is again based on condition (2.19) and the result is

$$b_I^a = -\sum_{c=1}^N (\Lambda_I \chi_c) S^{ca}. \tag{3.18}$$

Hence, substituting this result into Eq. (3.17) we obtain

$$\Lambda_I^*(z) = \left(\Lambda_I - \sum_{a,c=1}^N (\Lambda_I \chi_c) S^{ca} \mathbf{H}_a\right)(z), \qquad \forall z \in \Gamma'(m_c). \tag{3.19}$$

which can be written as $\Lambda_I^* = \{\Lambda_I^*, \}$ on $\Gamma'(m_c)$, where $\Lambda_I^* = \Lambda_I - \sum_{a,c=1}^{N} \{\Lambda_I, \chi_c\} S^{ca} H_a$ (i.e., the "star" variable [15] associated to Λ_I).

Since the structure constants of a Lie group depend only on parametrization [16], the commutation relations between the generators Λ_I^*, $I = 1,..., 10$, are obviously the same as those between the corresponding generators Λ_I, $I = 1,..., 10$. So that in the standard parametrization of $\mathfrak{P}$

$$(\varepsilon_I) = (\omega^{\mu\nu}, A^\nu), \qquad \omega^{\nu\mu} = -\omega^{\mu\nu}, \mu, \nu = 0, 1, 2, 3,$$

the infinitesimal generators are those $\mathbf{P}_\mu$, $\mathbf{J}_{\mu\nu}$ defined at the beginning of the present subsection and their commutation relations are

$$[\mathbf{P}_\mu, \mathbf{P}_\nu] = 0, \qquad [\mathbf{P}_\lambda, \mathbf{J}_{\mu\nu}] = \eta_{\lambda\nu}\mathbf{P}_\mu - \eta_{\lambda\mu}\mathbf{P}_\nu,$$
$$[\mathbf{J}_{\mu\nu}, \mathbf{J}_{\lambda\sigma}] = \eta_{\mu\lambda}\mathbf{J}_{\nu\sigma} + \eta_{\nu\sigma}\mathbf{J}_{\mu\lambda} - \eta_{\mu\sigma}\mathbf{J}_{\nu\lambda} - \eta_{\nu\lambda}\mathbf{J}_{\mu\sigma}. \tag{3.20}$$

The corresponding generators $\mathbf{P}_\mu^*$ and $\mathbf{J}_{\mu\nu}^*$ on $\Gamma'(m_c)$ satisfy the same commutation relations.

We can also look for the Lie brackets $[\Lambda_I^*, \mathbf{H}]$, $I = 1,..., 10$, where $\mathbf{H}$ is defined by Eq. (2.21), obtaining

$$[\Lambda_I^*, \mathbf{H}] = 0, \qquad I = 1,..., 10. \tag{3.21}$$

The derivation of this result is rather intricate and use must be made of the formula

$$\mathbf{H}_a S^{cb} = - \sum_{d,l=1}^{N} S^{cd} \{K_a, \{K_d, \chi_1\}\} S^{lb} \qquad \text{on } \Gamma'(m_c), \tag{3.22}$$

which can be easily obtained from the Eq. (2.22).

The commutation relation (3.21) is the local expression for the following:

C.3.5. COROLLARY. *The local one-parameter group* $\exp(\sigma\mathbf{H})$ ($\sigma \in \mathbb{R}$) *on* Γ' *transforms the realization* g^* *of* $\mathfrak{P}$ *on* $\Gamma_\lambda(m_c)$ *into the realization on* $\Gamma_{\lambda+\sigma}(m_c)$.

3.c. *The World Line Condition (WLC)* [14]

In order to describe the dynamics of a N-particle system by means of a CHS we must previously state what is the position of each particle when the system is represented by the phase space point $z \in \Gamma'$. Thus, $4N$ functions φ_a^μ on $\Gamma'(m_c)$ must be given defining the N position four-vectors. What is commonly taken in the CHS appeared in the literature as $\varphi_a^\mu = q_a^\mu$ on $\Gamma'(m_c)$. Nevertheless we are not going to consider this restriction here in order to have a little more general result.

If we let the N-particle system start from $z \in \Gamma'(m_c)$ then the world line of (a) in the Minkowsky space will be given by

$$\gamma_a(z) \equiv \{x_a^\mu(\sigma, z) = \varphi_a^\mu((\exp(\sigma\mathbf{H})z), \sigma \in \mathbb{R}\}. \tag{3.23}$$

Thus, if we want the N-particle system to be relativistic invariant, we must require the action of $\mathfrak{P}$ on $\Gamma'(m_c)$ to leave the world lines (3.23) invariant. That is, given $z \in \Gamma'(m_c)$ and $(\varepsilon_I) \in \mathfrak{P}$, the kworld lines obtained, either from z or $g^*(\varepsilon_I)z$, must be the same apart from the coordinates transformation on M_4 defined by $(\varepsilon_I) \in \mathfrak{P}$. This is equivalent to requiring that $\forall z \in \Gamma'(m_c)$, $\forall (\varepsilon_I) \in \mathfrak{P}$, $\forall \sigma \in \mathbb{R}$ there exist $\sigma_a(\sigma, z, \varepsilon_I)$, $a = 1,..., N$, such that

$$L_\nu^\mu(\varepsilon_I)(x_a^\nu(\sigma, z) - A^\nu(\varepsilon_I)) = x_a^\mu(\sigma_a, g^*(\varepsilon_I)z) \tag{3.24}$$

or, equivalently,

$$L_\nu^\mu(\varepsilon_I)(\varphi_a^\nu(\exp(\sigma H)z) - A^\nu(\varepsilon_I)) = \varphi_a^\mu(\exp(\sigma_a H) \circ g^*(\varepsilon_I)z). \tag{3.25}$$

We obviously have that

$$\sigma_a(\sigma, z, 0) = \sigma. \tag{3.26}$$

Due to the group property, condition (3.25) is equivalent to the one obtained by taking the derivative with respect to ε_I at $(\varepsilon_I = 0)$,

$$C_I^\mu + C_{I\nu}^\mu \varphi_a^\nu(\exp(\sigma H)z) = \left\{ \Lambda_I^* \varphi_a^\mu + \left(\frac{\partial \sigma_a}{\partial \varepsilon_I}\right)_{(\sigma, z, 0)} H \varphi_a^\mu \right\} (\exp(\sigma H)z), \tag{3.27}$$

where

$$C_I^\mu = -\left(\frac{\partial A^\mu}{\partial \varepsilon_I}\right)_0 \quad \text{and} \quad C_{I\nu}^\mu = \left(\frac{\partial L_\nu^\mu}{\partial \varepsilon_I}\right)_0. \tag{3.28}$$

Then, taking into account that $\exp(\sigma H)z \in \Gamma'$, we can write Eq. (3.27) as

$$C_I^\mu + C_{I\nu}^\mu \cdot \varphi_a^\nu - \Lambda_I^* \varphi_a^\mu = \sigma_{aI} \cdot H \varphi_a^\mu \quad \text{on} \quad \Gamma'(m_c), a = 1,..., N, I = 1,..., 10, \tag{3.29}$$

where

$$\sigma_{aI}(z) \equiv \left(\frac{\partial \sigma_a}{\partial \varepsilon_I}\right)(0, z, 0). $$

What we have proved at this point is stated in the following:

T.3.6. THEOREM. *Given a CHS and $4N$ functions φ_a^μ on $\Gamma'(m_c)$ defining the positions of the individual particles, the world lines (3.23) are Poincaré invariant in the sense of (3.24) if, and only if, $10 \times N$ functions σ_{aI} exist such that condition (3.29) is satisfied.*

As we have commented before, the CHS appearing in the literature commonly take

$\varphi_a^\mu = q_a^\mu$ on $\Gamma'(m_c)$. In that case, since the realization of $\mathfrak{P}$ on TM_4^N is the one generated by the functions P_μ and $J_{\mu\nu}$—see Eq. (2.16)—we have that

$$\Lambda_I q_a^\mu = C_I^\mu + C_{I\nu}^\mu \cdot q_a^\nu \qquad \text{on } T^*M_4^N.$$

Thus, introducing it into Eq. (3.29) yields

$$(\Lambda_I q_a^\mu - \Lambda_I^* q_a^\mu) = \sigma_{aI} \cdot H q_a^\mu \qquad \text{on } \Gamma'(m_c), a = 1,\dots, N. \tag{3.30}$$

And, taking (3.19) into account, it is finally equivalent to

$$\sum_{b,c=1}^N (\Lambda_I \chi_c) \cdot S^{cb} \cdot (H_b q_a^\mu) = \sigma_{aI} \cdot H q_a^\mu \qquad \text{on} \Gamma'(m_c) \ a = 1,\dots, N. \tag{3.31}$$

It must be stressed that conditions (3.29)—and also (3.31)—only consist of $3 \times 10 \times N$ equations instead of $4 \times 10 \times N$ as it could seem, because the remaining $10 \times N$ equations must be used to eliminate the undetermined functions σ_{aI}, $a = 1,\dots, N$, $I = 1,\dots, 10$.

Since the world line (3.23) must be time like and pointing to the future, we have to require the φ_a^μ's to satisfy $H\varphi_a^0 > 0$ on $\Gamma'(m_c)$. So that, we can obtain the σ_{aI}'s from the $10 \times N$ zero components of the equations (3.29) and, substituting them into the $3 \times 10 \times N$ spatial ones, we have

$$C_I^j + C_{I\nu}^j \cdot \varphi_a^\nu - \Lambda_I^* \varphi_a^j = (C_I^0 + C_{I\nu}^0 \cdot \varphi_a^\nu - \Lambda_I^* \varphi_a^0) \cdot (H\varphi_a^j) \cdot (H\varphi_a^0)^{-1} \quad \text{on } \Gamma' \tag{3.32}$$

and a similar expression for Eqs. (3.31).

It could seem at this point that the WLC (3.29) depends on all the fixations $\chi_a(z, \tau)$, $a = 1,\dots, N$. However, if the fixations are taken in the simple form (3.9) then, the WLC does not restrict the fixation $\chi_1(z, \tau) = h(z) - \tau$, but only the remaining ones: $\chi_A(z)$, $A = 2,\dots, N$.

Indeed, let us consider the set of fixations

$$\chi_1(z, \tau) = h(z) - \tau, \qquad \chi_A(z), \qquad A = 2,\dots, N,$$

which will provide the extended phase space $\Gamma'(m_c)$ and the vector field $\mathbf{H}$. Let us now consider another set:

$$\tilde{\chi}_1(z, \tau) = \tilde{h}(z) - \tau, \qquad \chi_A(z), \qquad A = 2,\dots, N,$$

so that $H\tilde{h} \neq 0$ on $\Gamma'(m_c)$. This new set of fixations will provide the same extended phase space but a different tangent vector $\tilde{\mathbf{H}}$. However, since the latter $\tilde{\mathbf{H}}$ will be proportional to the former $\mathbf{H}$, the set of world lines for particles defined by (3.23) with any one of these two sets of fixations will be the same apart from a reparametrization.

Hence, it is obvious that if the first set of fixations satisfies the WLC then, the second will too.

A rather tedious calculation yields

$$\tilde{\sigma}_{aI} = \sigma_{aI} \cdot (\mathbf{H}\tilde{h}) - \sum_{b,c-1}^{N} (\Lambda_I \chi_b) \cdot S^{bc} \cdot \mathbf{H}_c(h - \tilde{h}) \tag{3.33}$$

for the relationship between the functions σ_{aI} and $\tilde{\sigma}_{aI}$ appearing in both cases on the right-hand side of (3.29).

Thus, it is clear the unessential role of the clock fixation $\chi_1(z, \tau) = h(z) - \tau$ in the WLC and it should be therefore a desirable formulation only in terms of the $\chi_A(z)$, $A = 2,..., N$. We shall do it here for the case $\varphi_a^\mu \equiv q_a^\mu$ on $\Gamma'(m_c)$.

From Eq. (3.31) and taking (2.21) into account, we have

$$\sum_{B-2}^{N} \sum_{b=1}^{N} (\Lambda_I \chi_B) \cdot S^{Bb} \cdot (\mathbf{H}_b q_a^\mu) = \sigma_{aI}' \cdot \sum_{b-1}^{N} (\mathbf{H}_b q_a^\mu) \cdot S^{1b}, \tag{3.34}$$

where $\sigma_{aI}' = \sigma_{aI} - \Lambda_I \chi_1$.

The latter expression for the WLC must be understood as follows: "For any nonsingular S^{ab} solution of the linear system

$$\sum_{b-1}^{N} S^{ab} \cdot (\mathbf{H}_b \chi_D) = \delta_D^a \qquad \text{on } \Gamma'(m_c), a = 1,..., N, D = 2,..., N.$$

Eqs. (3.34) must be satisfied."

It is finally interesting to realize that, when the fixations χ_A, $A = 2,..., N$, are Poincaré invariant and the positions coordinates $\varphi_a^\mu = q_a^\mu$ on $\Gamma'(m_c)$ are taken, then the WLC automatically holds good.

3d. *Symplectic Structure*

In order to carry through the Dirac program we need to endow $\Gamma_\lambda(m_c)$ (resp. $\Gamma'(m_c)$) with a symplectic structure (resp. a contact structure) which must be invariant under the realization of $\mathfrak{P}$ defined in Subsection 3.b.

When we consider the injection mappings

$$\begin{array}{ccc} \Gamma'(m_c) & \xrightarrow{\ j\ } & T^*M_4^N \\ i_\lambda \Big\uparrow & \nearrow{j_\lambda} & , \qquad j_\lambda = j \circ i_\lambda, \\ \Gamma_\lambda(m_c) & & \end{array} \tag{3.35}$$

we can use the respective pull-back maps to relate the exterior algebras on each one of these manifolds in the oposite direction:

$$\begin{array}{ccc} \Lambda(\Gamma'(m_c)) & \xleftarrow{\ j^*\ } & \Lambda(T^*M_4^N) \\ i_\lambda \Big\downarrow & \nearrow{f_\lambda} & , \qquad j_\lambda^* = i_\lambda^* \circ j^*. \\ \Lambda(\Gamma_\lambda(m_c)) & & \end{array}$$

Thus, the former symplectic form $\Omega \in \Lambda^2(T^*M_4^N)$ given by Eq. (2.15) defines the 2-differential forms

$$\omega = j^*\Omega \in \Lambda^2(\Gamma'(m_c)), \qquad \omega_\lambda = j_\lambda^*\Omega \in \Lambda^2(\Gamma_\lambda) \tag{3.36}$$

in an unambiguous way.

The commutating between the pull-back map and the exterior derivative is a well-known result of differential geometry [17], namely,

$$\begin{aligned}
d(j^*\Omega)|_z &= j^*(d\Omega)|_{j(z)=z}, &\quad \forall z \in \Gamma'(m_c) \subset T^*M_4^N, \\
d(j_\lambda^*\Omega)|_z &= j_\lambda^*(d\Omega)|_{j_\lambda(z)=z}, &\quad \forall z \in \Gamma_\lambda(m_c) \subset T^*M_4^N.
\end{aligned} \tag{3.37}$$

Therefore, since Ω is closed (i.e., $d\Omega = 0$), ω and ω_λ are too:

$$d\omega = 0 \text{ on } \Gamma'(m_c) \qquad \text{and} \qquad d\omega_\lambda = 0 \text{ on } \Gamma_\lambda(m_c). \tag{3.38}$$

It must be realized that in Eqs. (3.37) and (3.38) the same symbol "d" has been used to denote the three different exterior derivatives, on $T^*M_4^N$, $\Gamma'(m_c)$ and $\Gamma_\lambda(m_c)$, respectively. We have taken, however, this freedom of notation to avoid a too sophisticated notation since no confusion can arise from it.

If we ask for the ranks of ω and ω_λ, then the following lemma is needed:

L.3.7. LEMMA. *Let* $(\mathfrak{M}, \Omega)$ *be a symplectic* 2n-*manifold and* $\mathfrak{B} \xrightarrow{j} \mathfrak{M}$ *a submanifold defined by* k *constraints* $\{\eta_\alpha, \alpha = 1,...,k\}$ *such that the rank of the Poisson brackets matrix* $\{\eta_\alpha, \eta_\beta\}$ *is* $2r \leqslant k\}$. *Thus, it turns out that*

$$\text{rank } j^*\Omega = 2 \cdot (n - k + r). \tag{3.39}$$

The proof is given in Ref. [20].

Considering then the differential forms $\omega \in \Lambda^2(\Gamma'(m_c))$ and $\omega_\lambda \in \Lambda^2(\Gamma_\lambda(m_c))$, the sets of constraints defining their respective manifolds and taking condition (2.19) into account, an immediate application of this lemma yields

$$\text{rank } \omega = \text{rank } \omega_\lambda = 6N. \tag{3.40}$$

Thus, it turns out from Eqs. (3.38) and (3.40) that ω_λ is a symplectic form on $\Gamma_\lambda(m_c)$.

Furthermore, the symplectic form $\omega_\lambda \in \Lambda^2(\Gamma_\lambda(m_c))$ is left invariant by the realization g^* of $\mathfrak{P}$ on $\Gamma_\lambda(m_c)$. Indeed, for any infinitesimal generator Λ_I^*, $I = 1,..., 10$, of g^* and taking (3.18) and (3.19) into account, we have that

$$i(\Lambda_I^*)\omega_\lambda = j_\lambda^* \left\{ i \left(\Lambda_I + \sum_{a=1}^N b_I^a \mathbf{H}_a \right) \Omega \right\} = j_\lambda^* \{-d\Lambda_I - b_I^a \, dH_a\}$$

and, since $j_\lambda^*(dH_a) = d(H_a \circ j_\lambda) = 0$, we obtain

$$i(\Lambda_I^*)\, \omega_\lambda = -j_\lambda^*(d\Lambda_I) = -d(\Lambda_I \circ j_\lambda), \tag{3.41}$$

which implies

(i) $\mathfrak{L}(\Lambda_I^*)\, \omega_\lambda = 0,\; I = 1,..., 10,$

(ii)   the generating function on $\Gamma_\lambda(m_c)$ for the infinitesimal Poincaré transformation associated to ε_I is $\Lambda_I \circ j_\lambda$.

4. A Priori Hamiltonian Predictive Systems

This framework corresponds to a special attempt to construct some Hamiltonian and covariant predictive relativistic mechanics (i.e., a realization of the full symmetry group $\mathfrak{G}_N$ on TM_4^N, the generators of which having the standard shape (2.6) and (2.14), plus a $\mathfrak{G}_N$-invariant symplectic form).

The HPS procedure consists of two steps: first, working out a canonical realization of $\mathfrak{G}_N$ on the cotangent space $T^*M_4^N$ endowed with the natural symplectic form (2.15): $\Omega = \sum_{a=1}^{N} dq_a^\mu \wedge dp_\mu^a$, and second, to find out a diffeomorphism $\Xi: T^*M_4^N \to TM_4^N$ such that the jacobian map Ξ^T transforms the infinitesimal generators of $\mathfrak{G}_N$ on $T^*M_4^N$ into the vector fields (2.6) and (2.14) on TM_4^N. The inverse pull-back $(\Xi^{-1})^*$ then maps Ω into a $\mathfrak{G}_N$-invariant symplectic form on TM_4^N.

Nevertheless, as we shall see later, this approach presents the very unpleasant feature that the final action of $\mathfrak{G}_N$ on TM_4^N, namely, the dynamics, is not visualized until the end of the process and is critically determined by both steps. Thus, it will be rather impossible to formulate any known physical interaction into this framework.

We start from the phase space $T^*M_4^N$ with the symplectic form $\Omega = \sum_{a=1}^{N} dq_a^\mu \wedge dp_\mu^a$ and the standard action of $\mathfrak{P}$ generated by the functions P_μ and $J_{\mu\nu}$ defined in Eq. (2.16).

Then, N Poincaré invariant functions H_a on $T^*M_4^N$ are provided such that

$$\{H_a, H_b\} = 0, \qquad a, b = 1,..., N.$$

The $10 + N$ vector fields:

$$\mathbf{H}_a \equiv \{H_a, \;\}, \qquad \mathbf{P}_\mu \equiv \{P_\mu, \;\}, \qquad \mathbf{J}_{\mu\nu} \equiv \{J_{\mu\nu}, \;\} \tag{4.1}$$

obviously generate a realization of $\mathfrak{G}_N$ on $T^*M_4^N$ leaving Ω invariant.

As we shall see immediately the problem of finding out the above-mentioned diffeomorphism from $T^*M_4^N$ into TM_4^N can be solved by giving the position $x_a^\mu(z)$ and velocity $\pi_a^\mu(z)$ of each particle, $a = 1,..., N$, when the state of the system is $z \in T^*M_4^N$.

In this approach the position functions x_a^μ are required to satisfy:

(i) $\mathbf{H}_{a'} x_a^\mu = 0, \qquad \forall\, a' \neq a,$ \hfill (4.2)

(ii) $\mathbf{P}_\mu x_a^\nu = -\delta_\mu^\nu, \qquad \mathbf{J}_{\rho\lambda} x_a^\mu = \delta_\rho^\mu \cdot x_{a\lambda} - \delta_\lambda^\mu \cdot x_{a\rho},$ \hfill (4.3)

which, for the sake of brevity, will be written hereafter as

$$\Lambda_I x_a^\mu = C_I^\mu + C_{I\nu}^\mu \cdot x_a^\nu, \qquad I = 1,..., 10. \tag{4.4}$$

The latter condition guarantees the good behaviour of x_a^μ under the action of $\mathfrak{P}$ and the former one states that the position of particle (a) is "moved" by the action of $\mathbf{H}_a$ only. That is,

$$x_a^\mu \left(\exp \left[\sum_b \tau_b \mathbf{H}_b \right] z \right) = x_a^\mu (\exp [\tau_a \mathbf{H}_a] z), \qquad z \in T^* M_4^N, \tag{4.5}$$

which means that the evolution of the system starting from some initial state $z \in T^* M_4^N$ yields a world line for each particle.

Assuming that Eqs. (4.2) and (4.3) have been solved and the position functions x_a^μ are known, we then define the velocities by

$$\pi_a^\mu \equiv \left(\frac{2H_a}{u_a^2} \right)^{1/2} \cdot \mathbf{H}_a x_a^\mu, \qquad \text{where} \quad u_a^2 \equiv \mathbf{H}_a x_{a\mu} \cdot \mathbf{H}_a x_a^\mu \tag{4.6}$$

and the accelerations by

$$\theta_a \equiv \left(\frac{2H_a}{u_a^2} \right)^{1/2} \cdot \mathbf{H}_a \pi_a^\mu. \tag{4.6'}$$

Provided that the functions x_a^μ have been properly chosen, the jacobian matrix $\partial (x_a^\mu, \pi_b^\nu)/\partial(q_a^\mu, p_\nu^b)$ is nonsingular on $T^* M_4^N$ (or, at least in an open subset of it) and we can then define the 1 to 1 mapping:

$$\begin{aligned} \Xi &: T^* M_4^N \to TM_4^N \\ z &\to \Xi(z) = (x_a^\mu(z), \pi_a^\nu(z)). \end{aligned} \tag{4.7}$$

Its jacobian map Ξ^T applies the vector fields $\mathbf{H}_a$, $\mathbf{J}_{\mu\nu}$ and $\mathbf{P}_\mu$ into:

$$\begin{aligned} \Xi^T \mathbf{H}_a^0 &= \left(\pi_a^\mu \cdot \frac{\partial}{\partial x_a^\mu} + \theta_a^\mu \cdot \frac{\partial}{\partial \pi_a^\mu} \right), \qquad \mathbf{H}_a^0 \equiv \left(\frac{2H_a}{u_a^2} \right)^{1/2} \cdot \mathbf{H}_a, \\ \Xi^T \mathbf{P}_\mu &= - \sum_{a=1}^N \frac{\partial}{\partial x_a^\mu}, \\ \Xi^T \mathbf{J}_{\mu\nu} &= \sum_{a=1}^N \left(x_{a\mu} \frac{\partial}{\partial x_a^\nu} - x_{a\nu} \frac{\partial}{\partial x_a^\mu} + \pi_{a\mu} \frac{\partial}{\partial \pi_a^\nu} - \pi_{a\nu} \frac{\partial}{\partial \pi_a^\mu} \right), \end{aligned} \tag{4.8}$$

which yields a covariant predictive relativistic mechanics on TM_4^N, since the generators $\Xi^T \mathbf{H}_a^0$, $\Xi^T \mathbf{P}_\mu$, $\Xi^T \mathbf{J}_{\mu\nu}$ have the right shape (2.6), (2.14) and the orthogonality of θ_a^μ and π_a^μ follows immediately from their definitions (4.6) and (4.6').

Going back to the problem of finding out solutions to Eqs. (4.2) and (4.3) we shall carry through the integration in two steps:

(i)   We shall first integrate Eqs. (4.2) for a suitable set of Cauchy data, namely, a submanifold Σ and $4N$ function $\varphi_a^\mu \in \Lambda^0(\Sigma)$, requiring

$$x_a^\mu(z_0) = \varphi_a^\mu(z_0), \qquad \forall\, z_0 \in \Sigma.$$

(ii)   Second, we shall use Eqs. (4.3) as conditions on the Cauchy data.

Since Eqs. (4.2) can be solved separately for each particle (a) and component (μ), and since the $(N-1)$ vector fields $\mathbf{H}_{a'}$, $a' \neq a$, commute with each other, we have that

$$\dim \Sigma = 7N + 1$$

and also that for any given $a = 1,..., N$, Σ must not be a characteristic submanifold of the remaining fields $\mathbf{H}_{a'}$, $a' \neq a$. The latter implies that, if Σ is defined by the $(N-1)$ independent functions Ψ_A, $A = 2,..., N$, then the $N \times (N-1)$ matrix $(\mathbf{H}_a \Psi_B)_{\substack{a = 1,...,N \\ B = 2,...,N}}$ has no vanishing $(N-1) \times (N-1)$ minors:

$$\forall\, a = 1,..., N, \qquad \det(\mathbf{H}_{a'} \Psi_B)_{\substack{a' \neq a \\ B = 2,...,N}} \neq 0 \text{ on } \Sigma. \tag{4.9}$$

This condition guarantees that, for any z (at least in the neighbourhood of Σ where the condition holds), the implicit function theorem can be applied and determines $N - 1$ real numbers $\overset{(a)}{\sigma}_{a'}(z)$ such that

$$\exp \left(\sum_{a' \neq a} \overset{(a)}{\sigma}_{a'}(z)\, \mathbf{H}_{a'} \right) z \in \Sigma. \tag{4.10}$$

Then, it immediately comes out that the solution of Eqs. (4.2) for the Cauchy data (Σ, φ_a^μ) is

$$x_a^\mu(z) = \varphi_a^\mu \left(\exp \left(\sum_{a' \neq a} \overset{(a)}{\sigma}_{a'}(z)\, \mathbf{H}_{a'} \right) z \right). \tag{4.11}$$

As has been mentioned before, Eqs. (4.3) are to be understood as condition on the initial data (Σ, φ_a^μ). The right behaviour of x_a^μ under Poincaré transformations requires that

$$x_a^\mu(g(\varepsilon_I) z) = L_\nu^\mu(\varepsilon_I) \cdot (x_a^\nu(z) - A^\nu(\varepsilon_I)) \tag{4.12}$$

for any $(\varepsilon_I) \in \mathfrak{P}$—or, at least, in a neighborhood of the identity $(\varepsilon_I) = (0)$.

Introducing expression (4.11) into Eq. (4.12), we obtain

$$\varphi_a^\mu \left(\exp \left(\sum_{a' \neq a} \overset{(a)}{\sigma}_{a'}(\varepsilon, z) \cdot \mathbf{H}_{a'} \right) \circ g(\varepsilon_I) z \right)$$

$$= L_\nu^\mu(\varepsilon_I) \cdot \left\{ \varphi_a^\nu \left(\exp \left(\sum_{a' \neq a} \overset{(a)}{\sigma}_{a'}(z) \cdot \mathbf{H}_{a'} \right) z \right) - A^\nu(\varepsilon_I) \right\}, \qquad (4.13)$$

where $\overset{(a)}{\sigma}_{a'}(\varepsilon_I, z) \equiv \overset{(a)}{\sigma}_{a'}(g(\varepsilon_I) z)$.

Then, since Σ satisfies condition (4.9), we have that, for any $z_0 \in \Sigma$, there exists a linear combination of $\mathbf{H}_b(z_0)$, $b = 1, \ldots, N$, being tangent to Σ, which is unique appart from a multiplicative coefficient. Thus, we choose any function Ψ_1 on Σ such that

$$\det(\mathbf{H}_a \Psi_b)_{a, b = 1, \ldots, N} \neq 0 \qquad \text{on } \Sigma. \qquad (4.14)$$

This chosen function Ψ_1 defines the following one-parameter family of submanifolds

$$\Sigma_\lambda \equiv \{ z \in \Sigma / \Psi_1(z) = \lambda \} \subset \Sigma.$$

The mathematical objects which we are now dealing with are pretty similar to those considered in Subsections 3b and 3c. Like there we shall also have in the present case:

(i)   a realization $g^*(\varepsilon_I)$ of $\mathfrak{P}$ on Σ leaving Σ_λ invariant and

(ii)   a unique vector field $\mathbf{H} = \sum_{c=1}^N b^c \cdot \mathbf{H}_c$ tangent to Σ such that $\mathbf{H} \Psi_1 = 1$.

Considering now the arguments of φ_a^μ on the left-hand side of Eq. (4.13), we shall have that

$$z_2 \equiv \exp \left(\sum_{a' \neq a} \overset{(a)}{\sigma}_{a'}(\varepsilon, z) \cdot \mathbf{H}_{a'} \right) \circ g(\varepsilon_I) z$$

will not lay in the same submanifold Σ_λ as

$$z_1 \equiv \exp \left(\sum_{a' \neq a} \sigma_{a'}(z) \cdot \mathbf{H}_{a'} \right) z,$$

i.e.,

$$\Psi_1(z_1) - \Psi_1(z_2) \equiv \lambda^a(z_1, \varepsilon_I) \neq 0. \qquad (4.15)$$

(Note that the latter difference should apparently depend on how far from Σ the point z is, however, it follows from the commutativity between $\exp(\sigma_b \cdot \mathbf{H}_b)$ and the Poincaré transformation $g(\varepsilon_I)$ that λ^a only depends on ε_I and z_1, i.e., the "trace" of z on Σ.)

The vector field $\mathbf{H}$ then permits bringing z_2 into $\hat{z}_2 \in \Sigma_{\lambda_1}$ $(\lambda_1 \equiv \Psi_1(z_1))$ via its exponential map

$$\hat{z}_2 \equiv \exp(\lambda^a(z_1, \varepsilon) \cdot \mathbf{H}) z_2 \qquad (4.16)$$

or, writing z_2 in terms of z_1,

$$\hat{z}_2 = \exp(\lambda^a(z_1, \varepsilon) \cdot \mathbf{H}) \circ \exp(\Sigma' \overset{(a)}{\sigma}_{a'}(\varepsilon, z) \cdot \mathbf{H}_{a'}) \circ g(\varepsilon_I)$$
$$\circ \exp(-\Sigma' \overset{(a)}{\sigma}_{a'}(z) \cdot \mathbf{H}_{a'}) z_1,$$

where Σ' stands for the summation over all indices $a' \neq a$. (See Fig. 2.)

Taking then into account the commutativity between the different mappings appearing on the right of the latter equation, it can be expressed as

$$z_2 = \exp\left(\sum_{b=1}^{N} \zeta_b \cdot \mathbf{H}_b\right) \circ g(\varepsilon_I) z_1,$$

which, by the uniqueness theorem (T.3.2), implies that $\hat{z}_2 = g^*(\varepsilon_I) z_1$ or equivalently, using Eq. (4.16),

$$z_2 = \exp(-\lambda^a(z_1, \varepsilon_I) \cdot \mathbf{H}) \circ g^*(\varepsilon_I) z_1. \qquad (4.17)$$
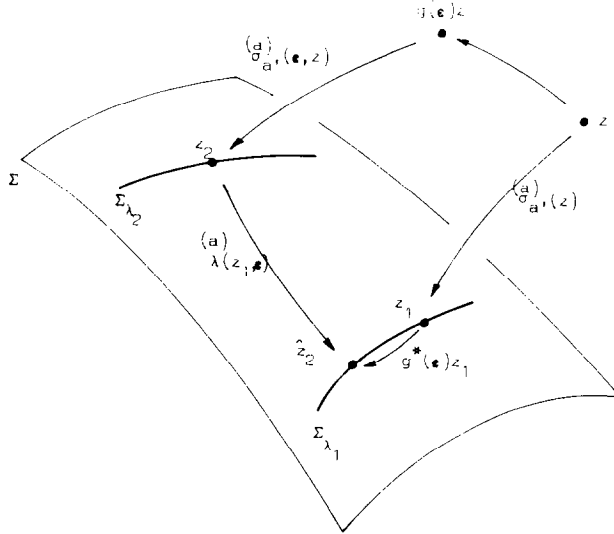

FIG. 2.   The relationship between the points z, $g(\varepsilon_I)z$, z_1, z_2 and $\hat{z}_2$ occurring in Eqs. (4.16), (4.17) and (4.18).

Using this expression together with the above definition for z_2 and z_1, Eq. (4.13) can be written as

$$\varphi_a^\mu \{\exp(-\lambda^a(z_1, \varepsilon) \cdot \mathbf{H}) \circ g^*(\varepsilon_I) z_1\} = L_\nu^\mu(\varepsilon) \cdot \{\varphi_a^\nu(z_1) - A^\nu(\varepsilon)\}, \qquad \forall z_1 \in \Sigma, \qquad (4.18)$$

which, derived relatively to ε_J at $(\varepsilon_I) = (0)$, yields the equivalent condition

$$C_J^\mu + C_{J\nu}^\mu \cdot \varphi_a^\nu = \overset{(a)}{\sigma}_J \cdot \mathbf{H}\varphi_a^\mu + \Lambda_J^* \varphi^\mu \qquad \text{on } \Sigma, \qquad (4.19)$$

where C_J^μ and $C_{J\nu}^\mu$ are defined in (3.28) and $\overset{(a)}{\sigma}_J(z_1) \equiv -\partial \lambda^a/\partial \varepsilon_J|_{(z_1, 0)}$.

So far we have proved that a set of Cauchy data (Σ, φ_a^μ) yields a solution of the position equations (4.2) which is well behaved under Poincaré transformations if, and only if, there exist $10 \times N$ functions $\overset{(a)}{\sigma}_J$ on Σ such that conditions (4.19) are satisfied. Realize that this result is quite similar to the world line condition for CHS. Indeed, Eq. (4.19) is identical to (3.29). Hence, the consequences commented there also hold in the present case.

5. EQUIVALENCE OF BOTH FORMALISMS: CHS AND HPS

In the CHS framework a given N-particle relativistic system can be always formulated in terms of $(2N - 1)$ functions, on the phase space $T^*M_4^N$:

$$K_a(z, m_a) = H_a(z) + \tfrac{1}{2}m_a^2, \qquad \chi_B(z), \quad a = 1,..., N, \quad B = 2,..., N,$$

such that

    (i) $\{H_a, H_b\} = 0$,

    (ii) $\text{rang}(\{H_a, \chi_B\})_{\substack{a = 1,...,N \\ B = 2,...,N}} = N - 1$ on   the   submanifold   defined   by
$H_a + \tfrac{1}{2}m_a^2 = 0$, $\chi_B = 0$, and

    (iii) the world line condition (3.30) is fulfilled.

Given these mathematical objects, we choose a suitable clock constant $\chi_1(z) = \tau$, we then work the formalism getting, at the end, a set of N world lines—defined by Eq. (3.23)—for each initial state z_0 on the extended phase space $\Gamma'(m_c)$.

On the other hand, a HPS is also formulated in terms of $(2N - 1)$ functions on $T^*M_4^N$: $H_a, \chi_B, a = 1,..., N, B = 2,..., N$ such that

    (i') $\{H_a, H_b\} = 0$,

    (ii') $(\{H_a, \chi_B\})_{a = 1,...,N, B = 2,...,N}$ does not have any vanishing $(N - 1) \times$
$(N - 1)$ minors, and

    (iii') the word line condition is fulfilled.

Working then all these objects in the HPS formalism we also obtain, at the end, a

set of N world lines for each initial state z belonging to the whole phase space $T^*M_4^N$. (Realize that the Cauchy data $\varphi_a^\mu = q_a^\mu$ are assumed in both cases, i.e., HPS and CHS.)

Thus, the same ensemble of mathematical objects (see footnote [21]) can be used in two different manners, namely, CHS and HPS, to obtain one relativistically invariant world line for each particle. What we are going to show throughout this section is that the same world lines are obtained in both frameworks.

T.5.1. THEOREM. *For any $z \in T^*M_4^N$, the N-submanifold*

$$S(z) \equiv \left\{ \exp\left(\sum_b \tau_b \cdot \mathbf{H}_b \right) z, \tau_a \in \mathbb{R} \right\}$$

cuts $\Gamma'(m_c)$ on one orbit of $\mathbf{H}$ (the masses appearing as parameters in $\Gamma'(m_c)$ are to be determined by $m_c^2 = -2H_c(z)$, $c = 1,\ldots, N$).

Proof. Since the matrix $(\mathbf{H}_b \chi_A)_{b=1,\ldots,N, A=2,\ldots,N}$ satisfies Eq. (4.9), it is always possible, for any fixed value of (a), to find out $(N-1)$ unique functions $\overset{a}{\sigma}_{a'}(\tau_a, z)$, $a' \neq a$, such that

$$\chi_B \left\{ \exp\left(\sum_{b=1}^N \overset{a}{\sigma}_b(\tau_a, z) \cdot \mathbf{H}_b \right) z \right\} = 0, \qquad B = 2,\ldots, N, \tag{5.1}$$

where $\overset{a}{\sigma}_a(\tau_a, z) \equiv \tau_a$. It obviously comes out that $\overset{a}{\sigma}_{a'}(0, z) = \overset{(a)}{\sigma}_{a'}(z)$, given by Eq. (4.10).

Thus, the curve

$$\left\{ \overset{a}{\gamma}(\tau_a, z) \equiv \exp\left(\sum_{b=1}^N \overset{a}{\sigma}_b(\tau_a, z) \cdot \mathbf{H}_b \right) z, \tau_a \in \mathbb{R} \right\} \tag{5.2}$$

is the intersection of $S(z)$ with $\Gamma'(m_c)$, and its tangent vector at any point $\overset{a}{\gamma}(\tau_a, z)$ can be written as

$$\sum_{b=1}^N \frac{\partial \overset{a}{\sigma}_b(\tau_a, z)}{\partial \tau_a} \mathbf{H}_b. \tag{5.3}$$

The uniqueness condition (2.19) then implies that this tangent vector is proportional to $\mathbf{H}$ at any point $\overset{a}{\gamma}(\tau_a, z)$. Hence, curve (5.2) is an orbit of $\mathbf{H}$ apart from a reparametrization.

It could still happen that each chosen value of (a) yields a different line $\overset{a}{\gamma}(\tau_a, \tau)$ on $\Gamma'(m_c)$. However, condition (4.9) again guarantees that all the coefficients on the right of Eq. (5.3) do not vanish and, therefore, all the different curves $\overset{a}{\gamma}(\tau_a, z)$, $a = 1,\ldots, N$, are equivalent (i.e., can be reparametrized into one another). Thus, we shall take a common parameter ρ, writing hereafter $\gamma(\rho, z) \equiv \overset{a}{\gamma}(\overset{a}{t}(\rho), z)$.

At this point everything is ready to compare both formalisms. For any given $z \in T^*M_4^N$, the world line $\{x_a^\mu(\exp[\tau_a \mathbf{H}_a]z), \tau_a \in \mathbb{R}\}$ is assigned to the particle (a) by the HPS formalism. Then, since x_a^μ is a solution of Eq. (4.2), we have that

$$x_a^\mu(\exp[\tau_a(\rho)\mathbf{H}_a]z) = x_a^\mu(\gamma(\rho, z)),$$

which, since $\overset{a}{\gamma}(\tau_a, z)$ lays on $\Gamma'(m_c)$, can also be written as

$$x_a^\mu(\exp[\tau_a(\rho)\,\mathbf{H}_a]z) = q_a^\mu(\gamma(\rho, z)).$$

Finally, taking into account that $\gamma(\rho, z)$ is an orbit of $\mathbf{H}$ on $\Gamma'(m_c)$ appart from a reparametrization $\rho = \rho(\lambda)$, we have

$$x_a^\mu(\exp[\tau_a(\lambda)\mathbf{H}]z) = q_a^\mu(\exp[\lambda\mathbf{H}]z_0), \qquad \lambda \in \mathbb{R},$$

$$\tau_a(\lambda) \equiv \tau_a(\rho(\lambda)) \qquad \text{and} \qquad z_0 \equiv \gamma(0, z) \in \Gamma'(m_c),$$
(5.4)

which proves that any N-tuple of world lines derived by the HPS formalism can be also obtained in the CHS framework.

That the converse (namely, that any N-tuple of world lines yielded by the CHS formalism can be derived as well by the HPS one) is also true follows immediately from Eq. (4.2) and from the fact that $\Gamma'(m_c)$ is a submanifold of the whole phase space $T^*M_4^N$.

6. Constrained Hamiltonian Systems CHS and Noncovariant Predictive Relativistic Mechanics PRM-3

Let a certain CHS be defined by the $2N$ constraints:

$$K_a(z, m) \equiv H_a(z) + \tfrac{1}{2}m_a^2, \chi_A(z), \chi_1(z, \tau) = n(z) - \tau,$$

$$a = 1, ..., N, A = 2, ..., N, z \in T^*M_4^N,$$

and let the particle positions in M_4 be given by some functions φ_a^μ on $\Gamma'(m_c)$. Besides, since each φ_b^0 ($b = 1, ..., N$) means a time and the vector field $\mathbf{H}$ generates the "τ-time evolution" on $\Gamma'(m_c)$, the assumption that $\mathbf{H}\varphi_b^0 > 0$ on $\Gamma'(m_c)$ is implicit in any CHS.

Then, the implicit function theorem can be applied and the equation

$$\varphi_a^0(\exp[\tau\mathbf{H}]z) = h(z), \qquad z \in \Gamma'(m_c), \tau \in \mathbb{R}, a = 1, ..., N,$$

can be solved in τ (at least for z belonging to a neighborhood of $\Gamma_0(m_c)$). Thus, for any z, there exists $\lambda_a(z) \in \mathbb{R}$ such that

$$\varphi_a^0(\exp[\lambda_a(z)\mathbf{H}]z) = h(z), \qquad a = 1, ..., N.$$

Using these N functions λ_a we can label $\Gamma'(m_c)$ by the coordinates

$$\tau = h(z), \tag{6.2a}$$

$$y_a^i(z) = \varphi_a^i(\exp[\lambda_a(z)\mathbf{H}]z), \tag{6.2b}$$

$$w_a^i(z) = \left(\frac{\mathbf{H}\varphi_a^i}{\mathbf{H}\varphi_a^0}\right) \exp[\lambda_a(z)\mathbf{H}]z. \tag{6.2c}$$

The "velocities" $w_a^i(z)$ so defined can also be expressed as

$$w_a^i = \mathbf{H}\,y_a^i. \tag{6.3}$$

Indeed, deriving along $\mathbf{H}$ Eq. (6.1), we obtain

$$(\mathbf{H}h)(z) = \mathbf{H}\{\varphi_a^0(\exp[\lambda_a(z)\mathbf{H}]z)\} = \{1 + (\mathbf{H}\lambda_a)(z)\}(\mathbf{H}\varphi_a^0)(\exp[\lambda_a\mathbf{H}]z).$$

Then, since, by definition of $\mathbf{H}$, $\mathbf{H}h = 1$, we have that

$$(\mathbf{H}\varphi_a^0)(\exp[\lambda_a\mathbf{H}]z) = \{1 + (\mathbf{H}\lambda_a)(z)\}^{-1}, \tag{6.4}$$

which can be used to calculate the derivative of y_a^i, yielding (6.3).

We can now express the eleven generators $\mathbf{H}$, Λ_I^*, $I = 1,..., 10$, in terms of these $(6N + 1)$ coordinates obtaining

$$\Lambda_I^* h = 0 \qquad \text{and} \qquad \mathbf{H}h = 1 \tag{6.5}$$

$$\Lambda_I^* y_a^i = C_I^i - C_I^0 w_a^i + y_a^j(C_{Ij}^i - C_{Ij}^0 w_a^i) + \tau C_{I0}^i, \tag{6.6}$$

$$\mathbf{H}y_a^i = w_a^i, \tag{6.3}$$

$$\Lambda_I^* w_a^i = w_a^j(C_{Ij}^i - C_{Ij}^0 w_a^i) + C_{I0}^i - (C_I^0 + y_a^j C_{Ij}^0)\mu_a^i, \tag{6.7}$$

$$\mathbf{H}w_a^i = \mu_a^i, \tag{6.8}$$

where the constants C_I^μ and $C_{I\nu}^\mu$ are given by Eqs. (3.28), and Eq. (6.7) defines the "acceleration" functions μ_a^i. (In the derivation of Eq. (6.6), the world line condition (3.29) and Eqs. (6.1) and (6.5) have been used. Afterwards, since $\mathbf{H}$ commutes with any Poincaré generator Λ_I^*, Eq. (6.7) has been obtained by deriving both sides of (6.6) along $\mathbf{H}$.)

In the standard parametrization of $\mathfrak{P}$, the constants C_I^μ and $C_{I\nu}^\mu$ are given by (4.3) and (4.4). So that, we can write

$$\mathbf{P}_0^* = \sum_{a=1}^{N} \left(w_a^i \frac{\partial}{\partial y_a^i} + \mu_a^i \frac{\partial}{\partial w_a^i}\right), \tag{6.9a}$$

$$\mathbf{P}_I^* = -\sum_{a=1}^{N} \frac{\partial}{\partial y_a^I}, \tag{6.9b}$$

$$\mathbf{J}_l^* = \tfrac{1}{2}\varepsilon_l{}^{rs}\mathbf{J}_{rs}^* = \sum_{a=1}^{N} \varepsilon_l{}^{ji} \left(y_{ai} \frac{\partial}{\partial y_a^j} + w_{al} \frac{\partial}{\partial w_a^j} \right), \tag{6.9c}$$

$$\mathbf{K}_l^* = \mathbf{J}_{l0}^* = -\tau \mathbf{P}_l^* + \sum_{a=1}^{N} \left[y_{al} w_a^i \frac{\partial}{\partial y_a^i} - (\delta_l^i - w_{al} w_a^i - y_{al} \mu_a^i) \frac{\partial}{\partial w_a^i} \right], \tag{6.9d}$$

$$\mathbf{H} = \frac{\partial}{\partial \tau} + \mathbf{P}_0^*. \tag{6.9e}$$

Furthermore, the commutation relation $[\mathbf{H}, \mathbf{P}_0^*] = 0$ implies that μ_a^i does not depend on τ explicitely, i.e.,

$$\mu_a^i = \mu_a^i(y_b^j, w_c^k).$$

We realize that expressions (6.9) for the Poincaré generators are almost identical to those given by Eqs. (2.4), which would be obtained for the noncovariant predictive relativistic system

$$\frac{dy_a^i}{dt} = w_a^i, \qquad \frac{dw_a^i}{dt} = \mu_a^i(y_b^j, w_c^k) \tag{6.10}$$

The only discordant thing is the term $-\tau\mathbf{P}_l^*$ appearing on the right-hand side of (6.9d). It is not surprising, in any case, that some disagreements arise between both sets of generators, respective (2.4) and (6.9). Because, whereas the former set acts on a $6N$-dimensional phase space (i.e., $T\mathbb{R}^{3N}$), the latter acts on the $(6N+1)$-dimensional extended phase space $\Gamma'(m_c)$. Thus, to compare them on equal footing we need either (i) to consider the restricted phase space $\Gamma_0(m_c) \subset \Gamma'(m_c)$ for the CHS (i.e., by making $\tau = 0$), where expressions (6.9) become fully equivalent to (2.4), or (ii) to implement $T\mathbb{R}^{3N}$ by one more variable ($t \in \mathbb{R}$) and extend the predictive relativistic system (2.4) to the whole extended phase space $T\mathbb{R}^{3N} \times \mathbb{R}$.

In the latter case, a t-time variation generator $\mathbf{T}$, commuting with the action of $\mathfrak{P}$ on $T\mathbb{R}^{3N} \times \mathbb{R}$, must be added and then many possibilities arise depending on how this $\mathbf{T}$ is defined. For instance, if $\mathbf{T} = \partial/\partial t$ is chosen, then the extension of the Poincaré generators (2.4) to the whole $T\mathbb{R}^{3N} \times \mathbb{R}$ is obtained by merely taking

$$\Lambda_l'(t, y_a^i, w_b^j) = \Lambda_l(y_a^i, w_b^j), \qquad I = 1,\dots, 10. \tag{6.11}$$

However, this trivial choice for the eleventh generator $\mathbf{T}$ is dynamically meaningless, since it does not contain any dynamical information at all.

If, on the contrary, we want the dynamical evolution of the system to be involved in the time generator $\mathbf{T}$, the choice must be

$$\mathbf{T}(t, y, w) \equiv \frac{\partial}{\partial t} + \mathbf{P}_0(y, w). \tag{6.12}$$

The commutation relations $[\mathbf{T}, \Lambda_I'] = 0$, $I = 1,\dots, 10$, then provide the way to pull the

10 Poincaré generators $\Lambda'_j(t, y, w)$ out from the $6N$-dimensional phase space ($t = 0$), where the $\Lambda'_j(0, y, w)$ are equal to the $\Lambda'_j(y, w)$ given by (2.4). Doing this, we obtain that

$$\mathbf{P}'_0(t, x, v) = \mathbf{P}_0(x, v), \qquad \mathbf{P}'_1(t, x, v) = \mathbf{P}_1(x, v),$$

$$\mathbf{J}'_1(t, x, v) = \mathbf{J}_1(x, v), \qquad \mathbf{K}'_1(t, x, v) = -t\mathbf{P}_1(x, v) + \mathbf{K}_1(x, v),$$

(6.13)

which is in full agreement with Eqs. (6.9).

Another interesting method to check the equivalence between CHS and PRM-3 would have consisted of labeling $\Gamma'(m_c)$ by the coordinates

$$t = h(z),$$
(6.14a)

$$\bar{y}^i_a(z) = \varphi^i_a(\exp[\bar{\lambda}_a(z)\mathbf{H}]z),$$
(6.14b)

$$\bar{w}^i_a(z) = \left(\frac{\mathbf{H}\varphi^i_a}{\mathbf{H}\varphi^0_a}\right)(\exp[\bar{\lambda}_a(z)\mathbf{H}]z),$$
(6.14c)

where $\bar{\lambda}_a(z)$ is defined by

$$\varphi^0_a(\exp[\bar{\lambda}_a(z)\mathbf{H}]z) = 0, \qquad a = 1,..., N,$$
(6.15)

instead of Eq. (6.1) in the case considered before.

Then, expressing the Poincaré generators in terms of these coordinates, we obtain

$$\mathbf{P}^*_0 = \sum_{a=1}^{N} \left\{ \bar{w}^i_a \frac{\partial}{\partial \bar{y}^i} + \bar{\mu}^i_a \frac{\partial}{\partial \bar{w}^i_a} \right\},$$
(6.16a)

$$\mathbf{P}^*_i = -\sum_{a=1}^{N} \frac{\partial}{\partial \bar{y}^i_a},$$
(6.16b)

$$\mathbf{J}^*_1 = \sum_{a=1}^{N} \varepsilon^i_{lk} \left\{ \bar{y}^k_a \frac{\partial}{\partial \bar{y}^i_a} + \bar{w}^k_a \frac{\partial}{\partial \bar{w}^i_a} \right\},$$
(6.16c)

$$\mathbf{K}^*_j = \sum_{a=1}^{N} \left\{ \bar{y}_{aj}\bar{w}^i_a \frac{\partial}{\partial \bar{y}^i_a} + (\bar{w}_{aj}\bar{w}^i_a + \bar{y}_{aj}\bar{\mu}^i_a - \delta^i_j) \frac{\partial}{\partial \bar{w}^i_a} \right\},$$
(6.16d)

which coincide with Eqs. (2.4).

In coordinates (6.14) we also have

$$\mathbf{H} = \partial/\partial t.$$
(6.17)

Thus, the vector field $\mathbf{H}$ (which generates the variations of the time t) has been emptied from any dynamical contents by merely choosing a suitable set of coordinates. Therefore, $\mathbf{H}$ does not represent anything dynamically new which has not yet been introduced by the Poincaré generator of time translations $\mathbf{P}^*_0$. The same also holds for the time parameter t.

Our claim is that all the dynamical features represented via the usual CHS method on the $(6N + 1)$-dimensional extended phase space $\Gamma'(m_c)$ can be pictured on the $6N$-dimensional phase space $\Gamma_0(m_c)$ as well the time evolution and the world lines would be obtained by the action of $\mathbf{P}_0^*$. The relationship between both pictures would be analogous to the one between the well-known Heisenberg and Schrödinger pictures in quantum mechanics.

Furthermore, what has been commented above can be used as an argument against a conjecture of Sudarshan, Mukunda, Goldberg [14], where an important role is assigned to the "eleventh generator" $\mathbf{H}$ in the circumvention of the no-interaction theorems [8] by the CHS formalism since, as has been here shown, nothing new is introduced by this generator. Although we agree with these authors that constrained Hamiltonian dynamics goes beyond the Dirac's program, meaning that, if the 10 Poincaré generating functions are written in terms of a set of canonical variables relatively to the Dirac bracket on $\Gamma'(m_c)$, then their expression will depend critically on the constraints, and presumably none of the three forms proposed by Dirac [1] will be recovered. We shall devote a forthcoming paper to discussing in detail how and why no-interaction is circumvented by the constrained Hamiltonian formalisms.

7. PREDICTIVE HAMILTONIAN SYSTEMS AND NONCOVARIANT PREDICTIVE RELATIVISTIC MECHANICS

For any fixed values of the masses $(m_1,..., m_N)$, let us now consider the $6N$-submanifold $\Sigma_0(m_c) \subset T^*M_4^N$ defined by

$$2H_a(z) = \pi_a^2(z) = -m_a^2; \qquad x_b^0(z) = 0; \qquad a, b = 1,..., N, \tag{7.1}$$

where $x_a^\mu(z)$ and $\pi_a^\mu(z)$ have been defined in (4.11) and (4.6).

We then label $\Sigma_0(m_c)$ by the following $6N$ coordinates: (i) the $3N$ positions $x_a^i(z)$ defined by (4.11) and (ii) the $3N$ "velocities":

$$v_a^i \equiv \frac{\pi_a^i}{\pi_a^0} = \frac{u_a^i}{u_a^0}, \tag{7.2}$$

where u_a^μ stands for $\mathbf{H}_a x_a^\mu$.

The $6N$ submanifold $\Sigma_0(m_c)$ is not preserved by the action of the full symmetry group $\mathfrak{G}_N$ on $T^*M_4^N$, however, in a similar way to the one discussed in Section 3b we can derive

(A) *A realization of $\mathfrak{P}$ on $\Sigma_0(m_c)$*

For any $(\varepsilon_I) \in \mathfrak{P}$ we consider: $g'(\varepsilon_I): \Sigma_0 \to \Sigma_0$ defined by

$$g'(\varepsilon_I)z \equiv \exp\left\{ \sum_{a=1}^N \sigma_a(z, \varepsilon) \cdot \mathbf{H}_a \right\} \circ g(\varepsilon_I)z, \tag{7.3}$$

where the $\sigma_a(z, \varepsilon)$ can be obtained by solving the equations

$$x_b^0 \left\{ \exp \left(\sum_{a=1}^{N} \sigma_a \cdot \mathbf{H}_a \right) \circ g(\varepsilon) z \right\} = 0, \qquad b = 1,..., N. \tag{7.4}$$

The uniqueness of these solutions σ_a, $a = 1,..., N$, is guaranteed by the implicit function theorem because

$$\det(\mathbf{H}_a x_b^0) = \det(\delta_{ab} \cdot u_a^0) = u_1^0 ... u_N^0 \neq 0. \tag{7.5}$$

Now, using fixations (7.1) and taking into account that the inverse matrix of $\mathbf{H}_c x_a^0 = u_a^0 \cdot \delta_{ac}$ is

$$S'^{cb} = \delta^{bc} \cdot \frac{1}{u_c^0}, \tag{7.6}$$

we have that the Poincaré generators (3.19) are

$$\Lambda_I' = \Lambda_I - \sum_{b=1}^{N} (\Lambda_I x_b^0) \cdot \frac{1}{u_b^0} \cdot \mathbf{H}_b$$

or, using Eqs. (4.4) and constraints (7.1),

$$\Lambda_I' = \Lambda_I - \sum_{b=1}^{N} (C_I^0 + C_{Ij}^0 \cdot x_b^j) \cdot \frac{1}{u_b^0} \cdot \mathbf{H}_b, \tag{7.7}$$

which, expressed in terms of coordinates (7.2), read

$$\mathbf{P}_0' = \sum_{a=1}^{N} \left(v_a^i \cdot \frac{\partial}{\partial x_a^i} + \alpha_a^i \cdot \frac{\partial}{\partial v_a^i} \right), \tag{7.8a}$$

$$\mathbf{P}_i' = \sum_{a=1}^{N} \frac{\partial}{\partial x_a^i}, \tag{7.8b}$$

$$\mathbf{J}_i' = \sum_{a=1}^{N} \varepsilon_{i\,k}^{\,j} \left(x_a^k \cdot \frac{\partial}{\partial x_a^j} + v_a^k \cdot \frac{\partial}{\partial v_a^j} \right), \tag{7.8c}$$

$$\mathbf{K}_j' = \sum_{a=1}^{N} \left(x_a \cdot v_a^i \cdot \frac{\partial}{\partial x_a^i} + (v_{aj} \cdot v_a^i + x_{aj} \cdot \alpha_a^i - \delta_j^i) \cdot \frac{\partial}{\partial v_a^i} \right), \tag{7.8d}$$

where

$$\alpha_a^i \equiv \frac{1}{u_a^0} \cdot \mathbf{H}_a(u_a^i / u_a^0). \tag{7.9}$$

That is, the realization of the Poincaré group on Σ_0 (which is locally isomorphic to

$T\mathbb{R}^{3N}$ via coordinates (7.2)) associated to the noncovariant predictive relativistic system

$$\frac{dx_a^i}{dt} = v_a^i, \qquad \frac{dv_a^i}{dt} = \alpha_a^i(x, v). \tag{7.10}$$

It is interesting to realize that this predictive relativistic system is quite equivalent to that obtained from the CHS framework, i.e., (6.10). The proof is sketched by the following reflexions: for any $z \in \Sigma_0(m_c)$, let us consider the state z_0 determined by the intersection of $\Gamma_0(m_c)$ with the world submanifold $S(z)$—see (T.5.1). The coordinates $y_a^i(z_0)$ assigned by (6.2) correspond to the space coordinates of particle (a) in M_4 when its time coordinate is made zero by going forwards or backwards along its own world line (recall that the right-hand side of Eq. (6.1) is $h(z_0) = 0$, and that φ_a^0 and x_a^0 coincide on $\Gamma'(m_c)$). On the other hand, the coordinates (7.2), $x_a^i(z)$ give, by definition, the space position of (a) on its world line when the time is $x_a^0(z) = 0$ (since $z \in \Sigma_0(m_c)$ and the constraints (7.1) then apply). Finally, since the same world lines are assigned either to z by the PHS formalism or to z_0 in the CHS framework, as has been proved in Section 5, we can conclude

$$x_a^i(z) = y_a^i(z_0). \tag{7.11a}$$

For the same reasons, we can also state for the velocities and accelerations that

$$v_a^i(z) = w_a^i(z_0) \qquad \text{and} \qquad \alpha_a^i(z) = \mu_a^i(z_0). \tag{7.11b}$$

(B) *A symplectic Form on* $\Sigma_0(m_c)$

Indeed, if we call $\xi \colon \Sigma_0(m_c) \to T^*M_4^N$ the natural injection, then its pull-back mapping provides a differential 2-form,

$$\omega_0' \equiv \xi^* \Omega \in \Lambda^2(\Sigma_0), \tag{7.12}$$

which is symplectic, since the Poisson brackets matrix $\{H_a, x_b^0\} = \delta_{ab} \cdot u_b^0$ is non-singular.

Furthermore, as has been discussed above in a similar situation (see Subsections 3b and 3d), ω_0' is invariant under the realization of $\mathfrak{P}$ on $\Sigma_0(m_c)$ generated by (7.9). Moreover, as has been also commented there, the Poisson bracket associated to ω_0' on $\Sigma_0(m_c)$ is the Dirac bracket relative to the second class constraints (7.1).

Thus, a predictive Hamiltonian system provides a canonical noncovariant predictive relativistic mechanics by the following rule:

(i) Solve the constraint equations (7.1) to obtain $2N$ among the variables in terms of the remaining $6N$ and substitute them into the standard expression (2.16) for the Poincaré generating functions.

(ii) The action of $\mathfrak{P}$ on $\Sigma_0(m_c)$—which is locally isomorphic to $T\mathbb{R}^{3N}$—is generated by the resulting functions on $\Sigma_0(m_c)$ via the corresponding Dirac bracket.

8. CONCLUSION

The main result achieved in the present article has been the proof of the complete dynamical equivalence between the a priori Hamiltonian predictive relativistic systems and the constrained Hamiltonian relativistic ones, meaning that, to describe a N particle relativistic system, both formalisms start from the same mathematical objects, fulfilling the same requirements—although assigning them different meanings: mass shells or Hamiltonian functions and fixations or Cauchy data, respectively— and, after working them by different methods and from clearly differentiated standpoints, the same world lines are obtained in both frameworks. As far as the CHS formalism deals with a $(6N + 1)$-dimensional phase space whereas the HPS framework does with the $8N$-dimensional TM_4^N, it could be said that the latter is a predictive extension of the former.

We have also proved that a given constrained Hamiltonian relativistic system can be formulated in terms of noncovariant predictive relativistic mechanics, by merely choosing a suitable set of coordinates in the $(6N + 1)$-dimensional extended phase space.

There has clearly appeared that the "eleventh generator" $\mathbf{H}$ does not contain any dynamical information which has not been previously introduced by the 10 Poincaré generators. Hence, there seems to be no conceptual need for introducing either an eleventh generator ($\mathbf{H}$) or a strange "time parameter" (τ) as a phase space variable, although both of them may be useful to simplify the calculations.

Our conjecture is that an equivalent framework could be established by fixing $2N$ second class constrains, with no dependence on any extra parameter τ there. The time evolution, and hence the world lines, would be provided by the time translation generator $\mathbf{P}_0$. The relationship between this viewpoint and the usual one in the CHS formalism would be similar to the relationship between the Heisenberg and Schrödinger pictures in quantum mechanics.

We have also studied the world line condition and further analyzed its content, following from there that it must be understood as $30 \times N$ conditions on the N mass shell constraints and the $(N - 1)$ τ-independent fixations, regardless of the clock fixation.

In our opinion, the WLC is something previous to the no Interaction theorems arising in other approaches to the relativistic dynamics of particles. We mean by this that a too restrictive choice for the fixations $\chi_2(z),...,\chi_N(z)$, could forbide, via the WLC, any interaction to occur. For instance, if the fixations $\chi_A = q_A^0 - q_1^0$, $A = 2,..., N$, are taken, the WLC then implies that no interaction is possible (this has been proved in Ref. |23| for the special case of two particles with some specific mass shall constraints, and we shall prove it for the general case in a forthcoming paper).

It is finally interesting to point out that, thanks to the comparison of both formalisms (i.e., CHS and HPS), we have found out and proved that the world line condition of CHS is also needed in the HPS framework, having here a different meaning—recall that it has appeared as a condition on the Cauchy data for the particle positions to guarantee their good behaviour under Poincaré transformations.

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21. Note that condition (ii′) is more restrictive than (ii). However, although the general theory of constrained Hamiltonian relativistic systems takes (ii) as one of its main assumptions, the particular models which we have quoted fulfill (ii′) in addition. Futhermore, it must be stressed that if a given CHS is formulated in the simple fashion given at the beginning of Section 5, and fulfills condition (ii), but not (ii′), then the very unpleasant fact will occur that the "Hamiltonian" H, which generates the evolution of the system on Γ', will not depend on some of the masses of the particles.

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