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On the transition to columnar convection

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Convection in a rotating annulus with no-slip sidewalls, stress-free ends, radial gravity, and sideways heating is considered. The transition from fully three-dimensional convection cells to Taylor columns with increasing rotation rate is studied and its dependence on the annulus parameters is established. © *1995 American Institute of Physics.*

I. INTRODUCTION

Convection in a rotating right circular cylinder has a number of geophysical and astrophysical applications. For example, when the fluid is heated from below and gravity acts downwards, the resulting system serves as a model of convection in the polar regions. This system has recently been the subject of detailed laboratory experiments,[1] as well as careful linear stability analysis.[2,3] In the present note we consider a model of convection in the equatorial regions consisting of a rotating annulus with gravity radially inwards and outwards heating. This case, like the case considered by Zhong *et al.*,[1] is relatively simple in that it remains barotropic. We do not consider the baroclinic case, i.e., the case in which gravity and the applied temperature gradient are in different directions.

The system considered here is the simplest possible. We choose no-slip fixed temperature boundary conditions at $r=r_1$ and $r=r_2$, and adopt free-slip thermally insulating boundary conditions at the top and bottom. Systems of this type have been studied by several authors, notably Busse,[4] who pointed out that the resulting equations admit solutions in the form of Taylor columns, i.e., solutions that are independent of the axial coordinate, and have no axial velocity. Since such solutions are in any case expected for large rotation rates for which the Taylor–Proudman theorem applies, subsequent studies have focused on this two-dimensional regime. Busse[4] has shown that the resulting two-dimensional steady convection columns undergo precession once slanted ends are introduced in order to model effects due to curvature of a sphere. Precession occurs because the slanted ends break the Taylor–Proudman constraint.

In this note we consider the case of horizontal ends, but consider low enough rotation rates that the mode that first sets in does not take the form of a Taylor column. That this should occur is intuitively clear: when the angular velocity Ω of the system vanishes the system is essentially Rayleigh–Bénard convection. The resulting system admits two kinds of modes, axisymmetric and columnar (or more generally a superposition of the two), whose wavelength is determined by the annulus width. By analogy with Rayleigh–Bénard convection in a rectangular container the rolls try to align themselves along the shorter direction. Owing to the curvature of the annulus one expects columns if $R \gtrsim L$, and axisymmetric rolls if $R \lesssim L$. Here L is the height of the annulus and R is its mean radius. In particular one expects, under appropriate conditions, to find a critical mode with nonzero axial wave number. We show below that this is indeed the case, and determine the location in parameter space of the transition from cells to columns. It should be noted that when the cells have a nonzero azimuthal as well as axial wave number they precess in the rotating frame, even though the ends of the annulus are horizontal. This is in contrast to the columns. Consequently precessing cells are expected at most for $R \lesssim L$, at least for small rotation rates, and the precession rate is then expected to be of order the rotation rate.

In the basic state heat is transported radially by thermal conduction. The resulting temperature distribution is given by $T(r)=\Delta T \ln r/\ln \eta$, where $\Delta T \equiv T_1 - T_2$ and η $(0<\eta<1)$ is the radius ratio r_1/r_2. The stability of this state is described by the following linearized equations, nondimensionalized relative to the thermal diffusion time across the gap:

$$\frac{1}{\sigma}\frac{\partial \mathbf{u}}{\partial t}+2\mathbf{\Omega}\times\mathbf{u}=-\nabla p+\text{Ra}\ \Theta\hat{\mathbf{r}}+\nabla^2\mathbf{u}, \tag{1a}$$

$$\frac{\partial \Theta}{\partial t}=-\frac{u}{r\ln\eta}+\nabla^2\Theta, \tag{1b}$$

$$\nabla\cdot\mathbf{u}=0, \tag{1c}$$

where $\mathbf{u}=(u,v,w)$ is the velocity field in (r,ϕ,z) coordinates and Θ denotes the departure of temperature from the (dimensionless) conduction profile $T(r)/\Delta T$. The parameters Ra and σ denote, respectively, the Rayleigh and Prandtl numbers. The dimensionless angular velocity $\mathbf{\Omega}=(\Omega_{\text{phys}}d^2/\nu)\hat{\mathbf{z}}$, where $d\equiv r_2-r_1$ is the gap width. The boundary conditions are given by

$$\mathbf{u}=\Theta=0 \quad \text{on} \quad r=r_1,r_2, \tag{2}$$

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=\frac{\partial \Theta}{\partial z}=0 \quad \text{on} \quad z=0,\beta. \tag{3}$$

Here $\beta \equiv L/d$ denotes the inverse aspect ratio of the annulus.

Equations (1)–(3) are solved numerically using an expansion of the form

$$\Theta(r,\phi,z,t)=e^{st}\sum_{lmn} T_{lmn}(r)e^{in\phi}\cos\frac{m\pi z}{\beta}, \tag{4}$$

where s is the (possibly complex) growth rate. Each mode is specified by a pair of integers (m,n) indicating its structure

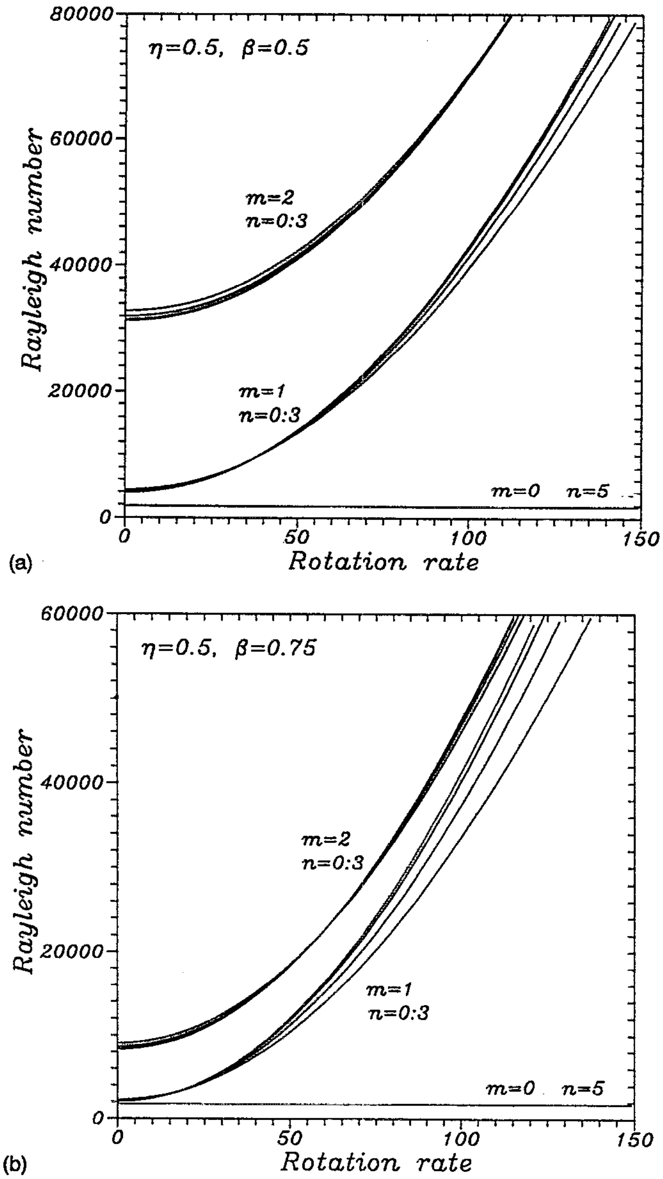


FIG. 1. The critical Rayleigh number $Ra_c^{(m,n)}$ as a function of the dimensionless rotation rate Ω for $\eta=0.5$ and (a) $\beta=0.50$, (b) $\beta=0.75$.

in the axial and azimuthal directions. The index l specifies the structure of the radial eigenfunction; in our calculations this structure is always the simplest possible, i.e., there are no nodes in the radial direction. The numerical code is related to that developed by Marqués *et al.*[5] and used by Goldstein *et al.*[2,3]

Equations (1)–(3) admit an exact solution of the form $u=u(r,\phi,t)$, $v=v(r,\phi,t)$, $w\equiv 0$, henceforth referred to as the Taylor column solution. In this case the incompressibility condition can be satisfied by introducing a streamfunction $\psi(r,\phi,t)$ such that

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial\phi},\quad v=\frac{\partial\psi}{\partial r}. \tag{5}$$

Equation (1a) can now be written in the form of a two-dimensional Rayleigh–Bénard problem

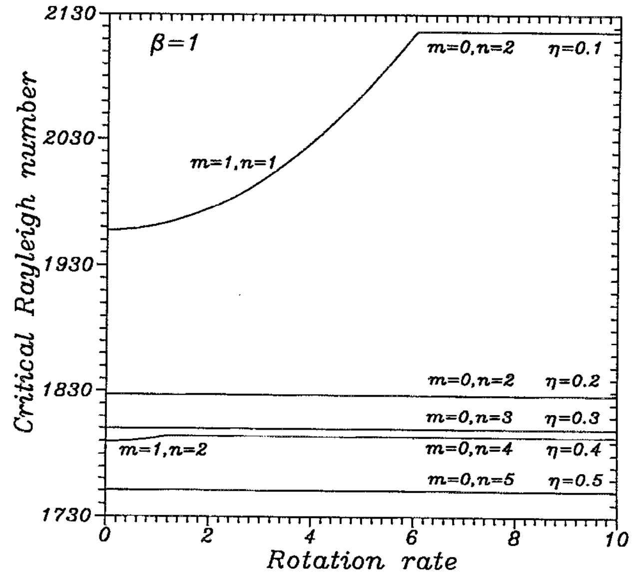


FIG. 2. As for Fig. 1 but showing $Ra_c^{(m,n)}$ for $\beta=1.0$ and several values of η.

$$\frac{1}{\sigma}\frac{\partial u}{\partial t}=-\frac{\partial P}{\partial r}+\mathrm{Ra}\,\Theta+\nabla^2\mathbf{u}\cdot\hat{\mathbf{r}}, \tag{6a}$$

$$\frac{1}{\sigma}\frac{\partial v}{\partial t}=-\frac{1}{r}\frac{\partial P}{\partial\phi}+\nabla^2\mathbf{u}\cdot\hat{\boldsymbol{\phi}}, \tag{6b}$$

where $P\equiv p-2\Omega\psi$. In this regime the Coriolis force can therefore be balanced entirely by an appropriate pressure gradient and the rotation Ω drops out. Consequently the bifurcation to Taylor columns will be a steady-state one. In the following we show that this $m=0$ solution is the first one that sets in if the rotation rate is sufficiently large; for smaller rotation $m\neq 0$ solutions are preferred.

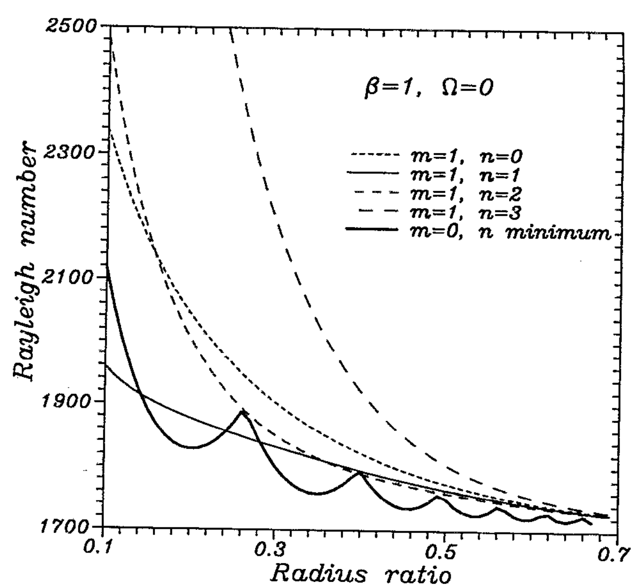


FIG. 3. The critical Rayleigh number $Ra_c^{(m,n)}$ as a function of the radius ratio η for $\beta=1.0$ and $\Omega=0$.

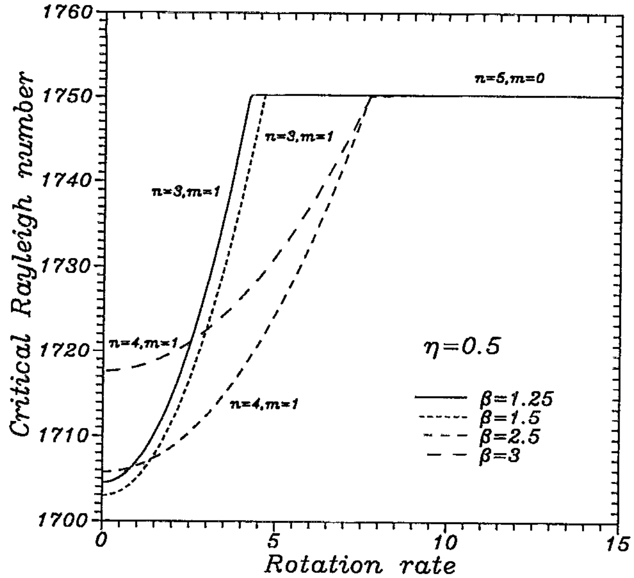
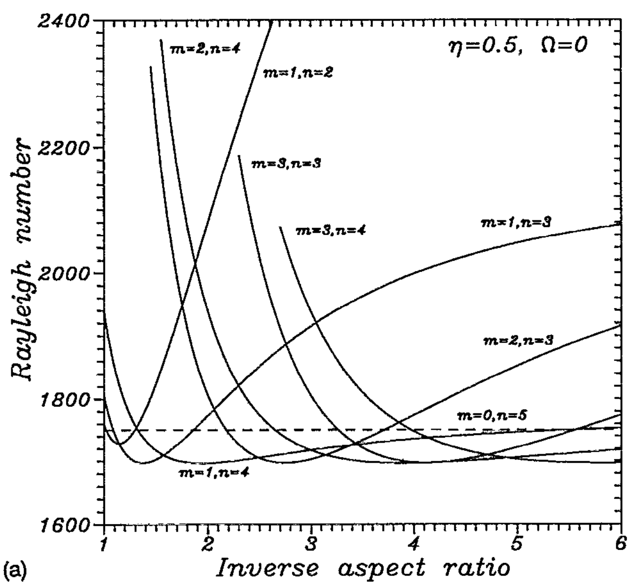


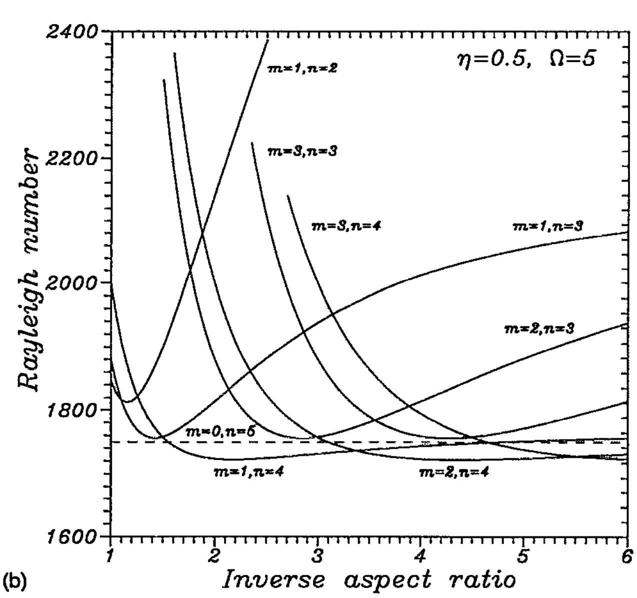
FIG. 4. The critical Rayleigh number $\mathrm{Ra}_c^{(m,n)}$ as a function of Ω for $\eta=0.5$ and several values of $\beta>1$.

II. RESULTS

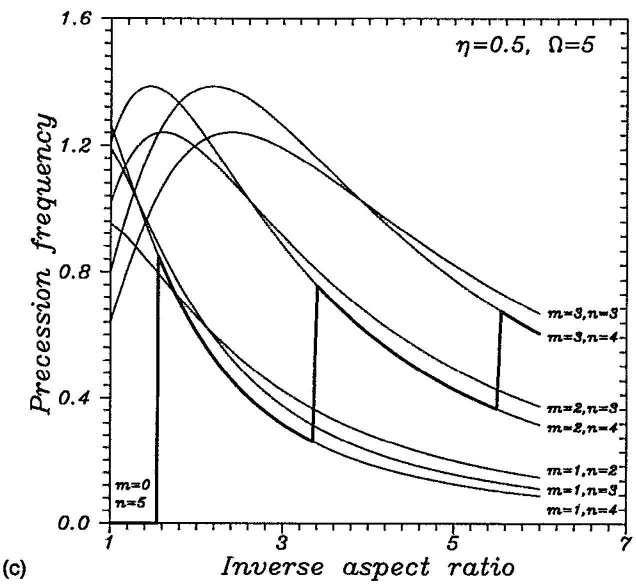
All the results reported below use $\sigma=6.7$, the Prandtl number of water. Results for infinite Prandtl numbers are expected to be nearly the same. As argued in the Introduction we expect to find Taylor columns even in the nonrotating system provided $L/R \lesssim 1$, i.e., when $\beta \lesssim \frac{1}{2}[(1+\eta)/(1-\eta)]$. Support for this argument is provided in Fig. 1 which shows the critical Rayleigh numbers Ra_c as a function of Ω for $m=0,1,2$ and $\beta=0.50$, 0.75. The radius ratio $\eta=0.5$. The different sets of curves correspond to different values of the azimuthal wave number n. The situation becomes more complicated when $\beta=1.0$. In Fig. 2 we show $\mathrm{Ra}_c(\Omega)$ for several different radius ratios. In each case only the lowest lying mode is shown. Observe that for some radius ratios (e.g., $\eta=0.2$, 0.3, 0.5) only Taylor columns are selected. As already explained the critical Rayleigh numbers for these modes are independent of Ω. For other radius ratios ($\eta=0.1$, 0.4) there is an interval of rotation rates in which the selected mode is an $m \neq 0$ mode. The origin of this behavior is explored in Fig. 3 which shows Ra_c as a function of η for $\beta=1.0$ and $\Omega=0$.



(a)



(b)



(c)

FIG. 5. The critical Rayleigh number $\mathrm{Ra}_c^{(m,n)}$ as a function of β for $\eta=0.5$ and (a) $\Omega=0$, (b) $\Omega=5$. The dashed lines represent the marginal curves for $m=0$. Panel (c) shows the precession frequencies corresponding to (b).

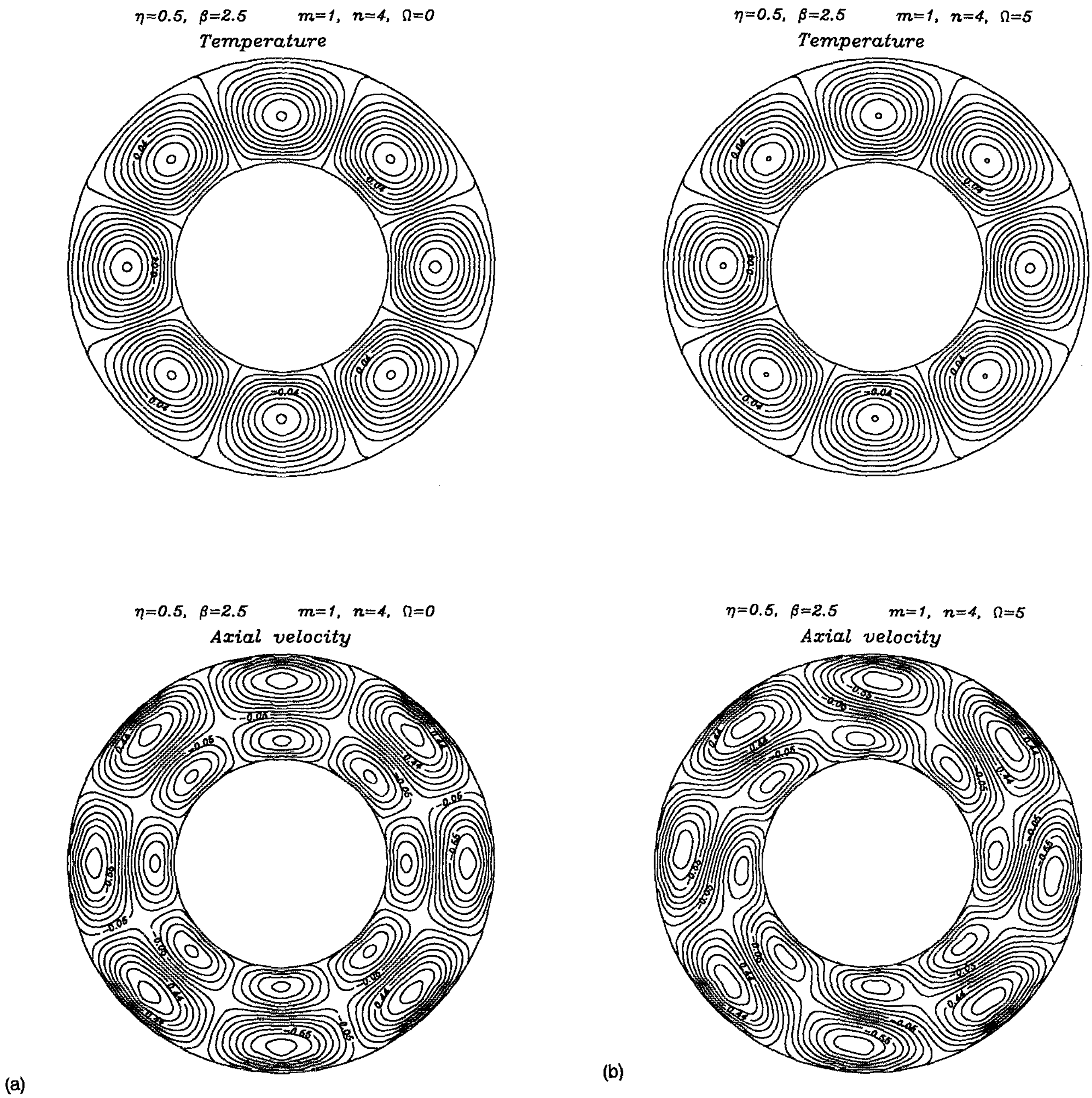


FIG. 6. The temperature (at $z=0$) and vertical velocity (at $z=\beta/2$) eigenfunctions for the mode $(m,n)=(1,4)$ with $\beta=2.5$, $\eta=0.5$ and (a) $\Omega=0$, (b) $\Omega=5$.

The threshold for Taylor columns with different values of n is shown by the heavy line. This line oscillates as a function of η, much as in the usual Rayleigh–Bénard problem, and for certain radius ratios crosses the locus of $m=1$ modes, either with $n=1$ or $n=2$. This trend continues and becomes more pronounced for larger β, as shown in Fig. 4. Note that for sufficiently low rotation rates the critical Rayleigh numbers for the $m\neq 0$ states are lower than that for the corresponding Taylor columns, and approach closely the usual Rayleigh–Bénard value. In contrast the lowest columnar mode has a somewhat higher Rayleigh number, as argued in the Introduction.

Finally, in Fig. 5 we show the dependence of Ra_c on β for $\eta=0.5$, and (a) $\Omega=0$, and (b) $\Omega=5$. Figure 5(c) shows the precession frequencies corresponding to Fig. 5(b). The corresponding eigenfunctions are shown in Fig. 6 for $\beta=2.5$, $\eta=0.5$, $\sigma=6.7$, $m=1$, $n=4$ and (a) $\Omega=0$, (b) $\Omega=5$. Both temperature and axial velocity eigenfunctions are shown; the latter illustrates dramatically the difference between the $m\neq 0$ and the $m=0$ solutions for which $w\equiv 0$. The parameter dependence of the transition from $(m,n)=(1,4)$ to $(0,5)$ is summarized in Fig. 7 which shows (a) the (Ω,η) plane for $\beta=2.5$, and (b) the (Ω,β) for $\eta=0.5$.

III. DISCUSSION

In this note we have shown that in a rotating annulus with radial gravity and sideways heating the dominant mode can be a precessing mode even though the boundary conditions assumed admit solutions in the form of nonprecessing

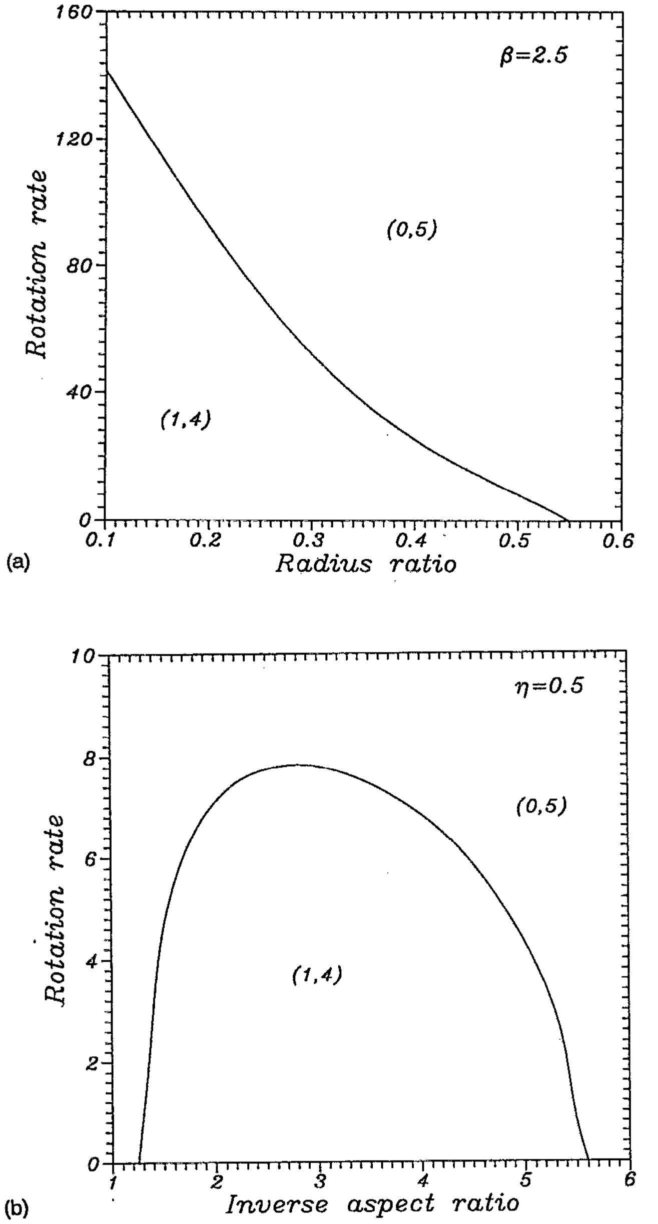


FIG. 7. The loci of the transition from $(m,n)=(1,4)$ to (0,5) in (a) the (Ω,η) plane for $\beta=2.5$, and (b) the (Ω,β) plane for $\eta=0.5$.

Taylor columns. We have examined the parameter dependence of the transition between these two types of modes and shown that it depends sensitively on both the annulus aspect ratio and its radius ratio. It is a simple matter to write down the form of the amplitude equations describing this transition in the nonlinear regime. In the following we write

$$\Theta(r,\phi,z,t)=A(t)f_1(r)e^{in_1\phi}+B(t)f_2(r)e^{in_2\phi} \times\cos(m\pi z/\beta)+\text{c.c}+\cdots \tag{7}$$

so that A is the (complex) amplitude of the Taylor columns and B is the amplitude of the cells, while $f_{1,2}$ are their radial eigenfunctions. In normal form[4] the equations for these amplitudes take the form

$$\dot{A}=\mu A+\alpha|A|^2A+\beta|B|^2A+\cdots, \tag{8a}$$

$$\dot{B}=\lambda B+\gamma|B|^2B+\delta|A|^2B+\cdots, \tag{8b}$$

where μ,α,β are real and λ,γ,δ are complex. The quantities μ,λ are unfolding parameters describing the location of the system in the parameter plane relative to the transition point (cf. Fig. 7), and are assumed to be small; the coefficients of the nonlinear terms can be computed at this point, provided certain nondegeneracy conditions hold, and depend on integrals over the radial and vertical eigenfunctions of the two competing modes. The resulting equations arise in a number of situations and their solutions are well known.[6,7] It follows that the transition from cells to columns in the nonlinear regime can take one of two forms: it can be hysteretic, or take place via a stable branch of mixed modes. By analogy with the Rayleigh–Bénard problem this transition is expected to depend sensitively on the Prandtl number, with high Prandtl numbers favoring the hysteretic transition (cf. Ref. 6). In a somewhat different context similar transitions have been studied in Refs. 8–11. Such nonlinear computations will be pursued elsewhere.

From the point of view of bifurcation theory the present problem is somewhat special. This is because one knows that generically in rotating systems a bifurcation that breaks the SO(2) symmetry of the system (i.e., has a nonzero azimuthal wave number) will take the form of a rotating wave.[12] This theorem fails for the Taylor columns because for these the Coriolis force has the form of a gradient, i.e., it is a nongeneric force. The argument suggests, however, that any physical effect that destroys the curl-free nature of the Coriolis force will lead to the appearance of precessing patterns. As shown by Busse[4] this can be accomplished by sloping ends; it can also be accomplished, perhaps more simply, by taking the boundary conditions at the top and bottom as rigid instead of stress-free. Such boundary conditions prevent the Taylor columns from being an exact solution of the governing equations and so will lead to patterns that precess at onset.

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