On boundary conditions for velocity potentials in confined flows: Application to Couette flow

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The representation of solenoidal fields by means of two scalar potentials can be a very useful method for a wide range of problems, in particular for the incompressible Navier–Stokes equations, though in finite containers boundary conditions may not be easily handled. The differential equations for the potentials are of an order higher than the original Navier–Stokes ones. As a consequence additional boundary conditions are needed to solve them. These differential equations and the corresponding boundary conditions for any geometry have been derived and the equivalence with the original problem has been proved. Special emphasis has been laid on domains with nontrivial geometry in which integral boundary conditions appear. As an example, the results have been applied to the periodic Couette flow. In this case the integral boundary conditions can be avoided by an appropriate change of variables, hence reducing the order of the equations obtained.

I. INTRODUCTION

The aim of the present paper is to implement the use of velocity potentials for the evolution equations of an incompressible viscous fluid in any geometry, for computing. These equations are the following:

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \mathbf{F} + \nu\Delta\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0, \tag{1}$$

where, if the force per unit of mass **F** depends on other variables such as temperature, magnetic fields, etc.,..., we must add the corresponding evolution equations. However, this will not alter the present analysis, which will be independent of the explicit form of **F**; for example, in convection problems the Boussinesq approximation leads to these types of equations. For convenience, we shall write Eqs. (1) in the form

$$(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b} = -\nabla p, \quad \mathbf{b} = \mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{F}, \quad \nabla\cdot\mathbf{v} = 0. \tag{2}$$

These equations must be complemented with initial and boundary conditions for **v**. The boundary conditions for rigid walls are of the form $\mathbf{v} = \mathbf{U}$, where **U** is the velocity of the boundary. We shall not treat the case where some of the boundary conditions are imposed on the pressure. In some cases these conditions can be transformed into conditions for the velocity (see Joseph,[1] p. 67). In this paper we shall only consider the case of three boundary conditions on **v**.

So as to eliminate both the pressure and the incompressibility condition $\nabla\cdot\mathbf{v} = 0$, several methods have been used. In Sec. II we discuss the toroidal and poloidal potentials:

$$\mathbf{v} = \nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e}), \tag{3}$$

where **e** is a vector field given beforehand. The Navier–Stokes equations are then replaced by the **e** components of their curl and double-curl and the pressure is hence eliminated. This method has been extensively used in thermal convection[2,3] and recently in Couette flow.[4]

In Sec. III we discuss the potential vector method $\mathbf{v} = \nabla\times\mathbf{B}$, with one of the components of **B** equal to zero:

$$\mathbf{v} = \nabla\times(\Psi^{(1)}\hat{\mathbf{e}}_x) + \nabla\times(\Psi^{(2)}\hat{\mathbf{e}}_y), \tag{4}$$

where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are orthogonal unit vectors; we take it in the x and y axis directions, respectively. Equations (1) are then replaced by the $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ components of the z derivative of their double-curl and the pressure is hence eliminated. This method has been used in thermal convection in a rectangular box.[5,6]

Both methods use two scalar potentials and give two equations of higher order (fourth or sixth depending on the variables considered). To solve these equations additional boundary conditions are needed. In the literature, explicit boundary conditions for the potentials have only been imposed partially, or in very simple situations. In many cases the additional boundary conditions needed are not explicitly stated and are hidden in the form chosen for the spectral expansions of the numerical methods used; and the authors do not prove in any circumstance the equivalence between the potential's equations (with the boundary conditions they use) and the initial Navier–Stokes problem (1). In the present paper we obtain these additional boundary conditions, proving the equivalence with the original problem; unfortunately for some of these conditions the potentials cannot be uncoupled. This is the price to be paid for using potentials. And this is probably the reason why some papers become unclear when displaying the boundary conditions.

When the fluid domain has a nontrivial geometry, i.e., when its homology groups are not trivial, integral boundary conditions appear. They are necessary to ensure that some differential closed forms are exact, allowing us to recover the Navier–Stokes equations from the ones written in terms of potentials. For that purpose we shall use the classical methods of differential geometry: the theory of homology, the deRahm theorem, and their applications to potential theory.[7–9]

In Sec. IV we apply the results obtained to the periodic Couette flow, which has a nontrivial geometry. Next we develop a reduction in the order of the equations obtained. This

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reduction makes clear the role of the integral conditions mentioned above, and gives a system of generalized parabolic equations [(41) and (44)], which are well suited for standard numerical integration techniques.

A different possibility for eliminating the pressure term is to apply the integral condition of the Galerkin method,[10] by using for the velocity vector a set of zero divergence basis functions. This is the equivalent of using two components of the curl of the Navier–Stokes equations. These methods, all of which use scalar potentials instead of the velocity itself, lead to equations of lower-order, coupling the two scalar potentials in the linear part of equations.

Another method for eliminating the pressure is the integrodifferential formulation of Achard and Canot.[11] They use potentials as intermediary tools in order to find a set of integrodifferential equations for the vorticity $\nabla\times\mathbf{v}$. This set is coupled to some Fredholm equations introducing four auxiliary variables defined in the boundary of the fluid domain.

The Clebsch potential method has also been used in fluid problems[12–14] and specifically in plasma physics.[15–17] The velocity field $\mathbf{v}$ is expressed via three potentials λ, μ, and ϕ in the following way:

$$\mathbf{v} = \lambda\nabla\mu + \nabla\phi; \tag{5}$$

λ and μ are taken such that the families of surfaces $\lambda = \text{const}$ and $\mu = \text{const}$ stratify the space, and the vortex lines coincide with these surface intersections. This formulation is useful in the Hamiltonian formulation of fluid and plasma systems. It has also been used to study turbulence and soliton structures. The three potentials λ, μ, and ϕ are coupled by the incompressibility condition $\nabla\cdot\mathbf{v} = 0$, and the μ potential is usually a multiple-valued function. The Clebsch potentials are rarely used in numerical calculations because of these facts, and we shall not deal with them in this paper.

II. TOROIDAL AND POLOIDAL POTENTIALS

In this formulation the velocity field $\mathbf{v}$ is expressed via the potentials ψ and ϕ as $\mathbf{v} = \nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e})$ [Eq. (3)], where $\mathbf{e}$ is a vector given beforehand. When the fluid is confined between two parallel planes as in the Bénard problem, or when the approximation of narrow gap in Couette flow is used (changing two concentric cylinders into two parallel planes), $\mathbf{e}$ is constant, orthogonal to the aforementioned planes. In problems concerning spherical symmetry (stellar convection[18]), one assumes $\mathbf{e} = \mathbf{r}$, the vector position. Both options lead to reasonably simple equations, and the potentials ψ and ϕ are associated, respectively, with the $\mathbf{e}$ components of vorticity and velocity.

In problems of cylindrical symmetry we could be induced by analogy to take $\mathbf{e} = (x,y,0)$, the radial vector. However, this gives rise to more complicated equations than in the previous cases. Furthermore, the association of the potentials with the vorticity and velocity is lost. Finally, in that case one cannot even assume the existence of the potentials ψ and ϕ. In such a simple case as that of the basic Couette flow they do not exist. See the particulars concerning these affirmations in Appendix A.

In the literature, explicit boundary conditions for the potentials[1,19] have only been imposed on surfaces orthogonal to $\mathbf{e}$, in which case they are very simple: $\psi = \phi = \mathbf{e}\cdot\nabla\phi = 0$. When the surfaces are not orthogonal to $\mathbf{e}$ the situation becomes more complicated, as we shall see. Thus potentials have often been avoided in closed containers.

We shall take $\mathbf{e}$ to be constant and unitary, although arbitrary geometries will be considered. We discuss first the existence and uniqueness of the potentials. Their conditions of existence in domains enclosed between parallel planes have been given by Joseph[1]; we shall show the existence of ψ and ϕ when the homology groups for the domain being considered are not trivial. Then we discuss in detail the boundary conditions for the potentials and the equivalence of both formulations (potentials versus primitive Navier–Stokes equations). The central result is a theorem, which gives us the additional boundary conditions (15) for the potentials. These conditions depend on the geometry of the fluid domain and include integral boundary conditions if the homology groups of the domain are not trivial.

Homology theory will be used at large in this paper, so we feel it may be wise to recall some definitions and examples for readers who are not familiar with them. Let Ω be an open domain of $\mathbb{R}^3$. The homology group $H_1(\Omega)$ is made of all classes of closed curves that are not boundaries. The closed curves that are boundaries of surfaces contained in Ω constitute the zero class. Two curves belong to the same class if both of them together define the boundary of a surface contained in Ω. For example, a connected domain $D\in\mathbb{R}^2$ with k holes has $H_1(D) = \mathbb{R}^k$; i.e., there are k nontrivial classes of closed curves, each of them constituted by curves surrounding one of the holes. The homology group $H_2(\Omega)$ is formed analogously by classes of closed surfaces. Two closed surfaces belong to the same class if both of them together are the boundary of a three-dimensional domain in Ω.

We list some useful properties and examples of the homology groups. If Ω is contractible to a point, all homology groups are trivial. If Ω is simply connected, $H_1(\Omega) = \{0\}$ is trivial. If Ω is a spherical shell, $H_1(\Omega) = \{0\}$ and $H_2(\Omega) = \mathbb{R}$. If Ω is a cylindrical shell or the interior of a torus, $H_1(\Omega) = \mathbb{R}$ and $H_2(\Omega) = \{0\}$. If Ω is a toroidal shell, $H_1(\Omega) = \mathbb{R}^2$ and $H_2(\Omega) = \mathbb{R}$. Periodic conditions identify different parts of $\partial\Omega$ giving rise to more complex geometries. Thus a rectangular box periodic in one direction gives rise to a cylindrical shell and double periodicity gives rise to a toroidal shell (see Fig. 1).

Let us first introduce the notations to be used. Let Ω be an open domain of $\mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$. We shall take the Cartesian z axis in direction $\mathbf{e}$, D_z as the intersection of Ω with the plane orthogonal to $\mathbf{e}$ at point z, and we shall call the unitary vectors tangent and normal to ∂D_z, respectively, $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ (see Fig. 2).

Classical potential theory tells us that a solenoidal vector field $\mathbf{v}$ derives from a potential vector $\mathbf{B}(\mathbf{v} = \nabla\times\mathbf{B})$ iff

$$\int_S \mathbf{v}\cdot d\,\mathbf{S} = 0 \quad \forall S\in H_2(\Omega), \tag{6}$$

where $H_2(\Omega)$ is the second homology group of Ω previously mentioned. The following proposition tells us that the condi-

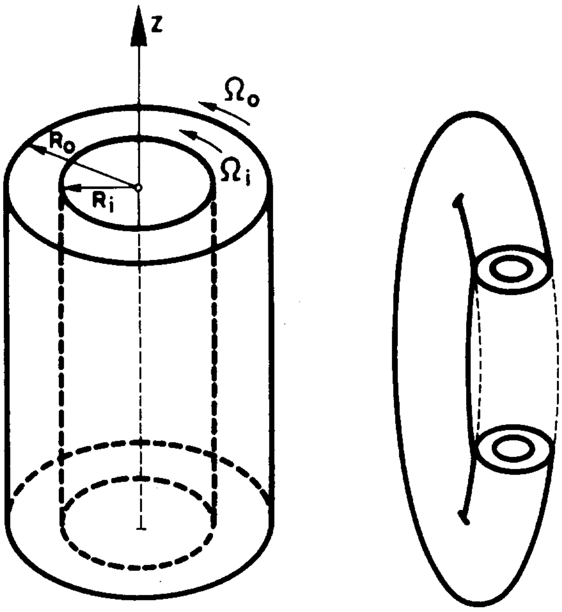

FIG. 1. Sketch of the Couette flow domain. By identifying the top and bottom of the cylinder, we obtain a toroidal shell.

tion of existence for the potentials ψ and ϕ coincides with (6).

Proposition 1: Let $\mathbf{v}$ be a continuous field of solenoidal vectors in Ω that verifies (6). Then there exist functions ψ and ϕ defined in Ω such that

$$\mathbf{v} = \nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e}). \tag{7}$$

Proof: The conditions (6) guarantee the existence of a vector potential $\mathbf{B}$:$\mathbf{v} = \nabla\times\mathbf{B}$. Let us consider the following Neumann problem in D_z:

$$\Delta_h f = -\partial_x B_x - \partial_y B_y \quad \text{in } D_z, \tag{8a}$$

$$\frac{df}{dn} = -\hat{\mathbf{n}}\cdot\mathbf{B} \quad \text{on } \partial D_z, \tag{8b}$$

where the subscript h (horizontal) refers to the Laplace operator in the plane domain D_z with respect to coordinates (x,y). If f is the solution to the Neumann problem (8), then the differential form

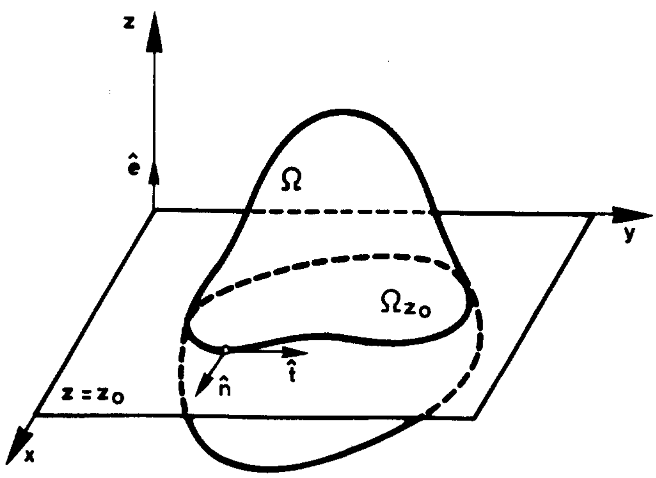

FIG. 2. Domains under consideration and notation.

$$\alpha = -(B_y + \partial_y f)dx + (B_x + \partial_x f)dy \tag{9}$$

is exact because of (8), as can be seen by direct calculation,

$$d\alpha = (\Delta_h f + \partial_x B_x + \partial_y B_y)dx\wedge dy = 0, \tag{10a}$$

$$\int_c \alpha = \int_c \left(\hat{\mathbf{n}}\cdot\mathbf{B} + \frac{df}{dn}\right) dl = 0, \quad \forall c \in H_1(D_z), \tag{10b}$$

since c always amounts to no more than a combination of the circuits that form ∂D_z. Now, there exists a function ϕ such that $d\phi = \alpha$. Coordinate z plays the role of a parameter in ∂D_z; if $\mathbf{B}$ is differentiable in z the solution to (8) and ϕ can be taken to be differentiable in z. An elementary calculation tells us that $\nabla\times(\phi\mathbf{e}) = \mathbf{B} + \nabla f - \psi\mathbf{e}$, where $\psi = B_z + \partial_z f$. Finally,

$$\mathbf{v} = \nabla\times\mathbf{B} = \nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e}), \tag{11}$$

and the proof is complete.

This result might seem trivial but there is no reason why it must be satisfied for a different choice of $\mathbf{e}$ (see Appendix A for a counterexample). We ourselves suppose that condition (6) is a consequence of the boundary conditions of $\mathbf{v}$ on $\partial\Omega$, as in fact happens in many specific cases. In fact, (6) guarantees that the potential vector $\mathbf{B}$ is single valued. If (6) is not satisfied, ψ and ϕ may still exist, but they will be multivalued functions. This is not a big deal because $\mathbf{v}$ remains single valued, but can make more difficult any numerical approach to the problem. If (6) is not satisfied, we can take $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, where $\mathbf{v}_0$ is a known velocity field satisfying $\nabla\cdot\mathbf{v}_0 = 0$, $\int\mathbf{v}'\cdot d\mathbf{S} = 0$. Then we can reformulate the problem in terms of the shifted velocity $\mathbf{v}'$, which now satisfies (6). See Appendix A for an example.

We may now examine the freedom in the choice of potentials (gauge freedom). An elementary calculation shows the following proposition.

Proposition 2: The solution to the homogeneous problem $\nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e}) = 0$ is given by

$$\Delta_h\phi = 0, \quad \int_c \frac{d\phi_z}{dn} dl = 0, \quad \forall c\in H_1(D_z), \tag{12a}$$

$$\nabla_h\psi = -\mathbf{e}\times\nabla\phi_z, \tag{12b}$$

where $H_1(D_z)$ is the homology group for closed curves on D_z defined as usual. So, therefore ϕ is determined up to a horizontal harmonic function [that is to say, harmonic in (x,y) and arbitrary in z]; and if $H_1(D_z)$ is not trivial the said function will have additional restrictions. With regard to ψ we can add only an arbitrary function of z to it (ψ is determined by ϕ in D_z up to an additive constant).

We shall now study the differential equations for ψ and ϕ. Given that the z components of the successive curls of $\mathbf{v}$ have a very simple expression,

$$\mathbf{e}\cdot\mathbf{v} = -\Delta_h\phi, \quad \mathbf{e}\cdot\nabla\times\mathbf{v} = -\Delta_h\psi, \quad \mathbf{e}\cdot\nabla\times\nabla\times\mathbf{v} = \Delta\Delta_h\phi, \tag{13}$$

we take the z components of the curl and double-curl of the Navier–Stokes equations (2), which gives

$$(\partial_t - \nu\Delta)\Delta_h\psi = \mathbf{e}\cdot\nabla\times\mathbf{b}, \tag{14a}$$

$$(\partial_t - \nu\Delta)\Delta\Delta_h\phi = -\mathbf{e}\cdot\nabla\times\nabla\times\mathbf{b}, \tag{14b}$$

so we have eliminated both the pressure and the $\nabla\cdot\mathbf{v} = 0$ condition, which is identically satisfied by (7). Since the equations obtained are of higher order than the initial ones, we must add additional boundary conditions so that problems (2) and (14) become equivalent. These additional conditions are given by the following theorem.

Theorem 1: Equations (2) and (14) are equivalent if we add the following boundary conditions to (14):

$$\hat{\mathbf{n}}\cdot\nabla\times[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}] = 0 \quad \text{on } \partial\Omega, \tag{15a}$$

$$\int_c [(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}]\cdot d\mathbf{l} = 0, \quad \forall c \in H_1(\Omega), \tag{15b}$$

$$\int_c \nabla\times[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}]\cdot d\mathbf{l} = 0, \quad \forall c \in H_1(D_z). \tag{15c}$$

Equivalence means that each solution to (14) and (15) defines a velocity $\mathbf{v} = \nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e})$ such that $(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}$ is a gradient; thus it defines the pressure p up to an additive constant and (2) can be recovered. Reciprocally, each smooth enough solution $\mathbf{v}$ of (2) defines some potentials ψ and ϕ, which satisfy (14) and (15). Obviously, if (2) is to be a well-posed problem, it is essential to give some boundary conditions for $\mathbf{v}$ on $\partial\Omega$, plus some initial conditions for $\mathbf{v}$ in $t = 0$ over Ω. The smoothness of solution $\mathbf{v}$ will depend upon the regularity of $\partial\Omega$ and on the boundary and initial conditions.[20] We ourselves shall assume that $\mathbf{v}$ solutions are sufficiently smooth (at least C^4, in order to be able to take the curl twice). The initial and boundary conditions for $\mathbf{v}$ ought to be considered as part of the problem (14). We shall now proceed to prove the theorem.

First, we shall demonstrate the equivalence between (2) and

$$(\partial_t - \nu\Delta)\nabla\times\mathbf{v} + \nabla\times\mathbf{b} = 0, \quad \nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{16a}$$

$$\int_c [(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}]\cdot d\mathbf{l} = 0, \quad \forall c \in H_1(\Omega), \tag{16b}$$

which is an immediate consequence of the following well-known property in potential theory:

$$\nabla\times\mathbf{A} = 0, \quad \int_c \mathbf{A}\cdot d\mathbf{l} = 0 \quad \forall c \in H_1(\Omega) \Leftrightarrow \mathbf{A} = -\nabla p. \tag{17}$$

Therefore it only remains to prove that (16) is equivalent to (14) and (15). This is a consequence of the following lemma.

Lemma 1: If $\mathbf{A}$ is a solenoid vector field of vectors in Ω, then $\mathbf{A} = 0$ in Ω is equivalent to

$$\mathbf{e}\cdot\mathbf{A} = 0, \quad \mathbf{e}\cdot\nabla\times\mathbf{A} = 0 \quad \text{in } \Omega, \tag{18a}$$

$$\hat{\mathbf{n}}\cdot\mathbf{A} = 0 \quad \text{on } \partial\Omega, \quad \int_c \mathbf{A}\cdot d\mathbf{l} = 0, \quad \forall c \in H_1(D_z). \tag{18b}$$

The direct implication is immediate. To prove the converse, we define the one-form in D_z as $\alpha = A_x\, dx + A_y\, dy$. The conditions

$$\begin{aligned} &\mathbf{e}\cdot\nabla\times\mathbf{A} = 0 \Rightarrow d\alpha = 0;\\ &\int_c \mathbf{A}\cdot d\mathbf{l} = 0 \Rightarrow \int_c \alpha = 0, \quad \forall c \in H_1(D_z), \end{aligned} \tag{19}$$

tell us that α is exact; hence there exists a function β such that $\alpha = d\beta$. We can see that β is constant in D_z. From (18) and the solenoidal character of $\mathbf{A}$,

$$\Delta_h \beta = \nabla\cdot\mathbf{A} = 0 \quad \text{in } \Omega, \tag{20a}$$

$$\frac{d\beta}{dn} = \hat{\mathbf{n}}\cdot\mathbf{A} = 0 \quad \text{on } \partial\Omega. \tag{20b}$$

Then β is the solution to the homogeneous Neumann problem in D_z and is therefore constant. As a consequence $\alpha = d\beta = 0$; hence $\mathbf{A} = 0$, which proves Lemma 1.

We are now able to complete the proof of Theorem 1. It is enough to apply Lemma 1 to $\mathbf{A} = \nabla\times[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}]$.

The integral boundary conditions [(15b) and (15c)] appear only if the homology groups $H_1(\Omega)$ and $H_1(D_z)$ are not trivial. If Ω and D_z are simply connected, only (15a) will remain as an additional boundary condition.

In order to integrate (14) we need five boundary conditions on $\partial\Omega$. We already have four, the three resulting from the boundary conditions of $\mathbf{v}$ plus (15a); the fifth one corresponds to the gauge freedom in the choice of potentials. We will choose ϕ or its normal derivative to be zero on $\partial\Omega$. If we solve (14) with additional boundary conditions different from (15) we must explicitly show that with these new conditions we can recover the Navier–Stokes equations (2). If some of the conditions (15) are lost, we may obtain as a solution of (14) velocity fields that satisfy the Navier–Stokes equations with additional terms. Their effect is the same as applying external forces, therefore modifying the dynamics of the fluid. See an example in Sec. IV.

III. THE POTENTIAL VECTOR METHOD

In this formulation the velocity field $\mathbf{v}$ is expressed as $\mathbf{v} = \nabla\times\mathbf{B}$, taking one of the components of vector $\mathbf{B}$ equal to zero: $\mathbf{v} = \nabla\times(\Psi^{(1)}\hat{\mathbf{e}}_x) + \nabla\times(\Psi^{(2)}\hat{\mathbf{e}}_y)$ [Eq. (4)]. We have chosen the z component of $\mathbf{B}$ equal to zero. Different authors[5,6] take these or other components to be zero. The results obtained in this paper work in both cases. The notations of this section are the same as in Sec. II, but we restrict ourselves to the case $\Omega = D\times(z_1,z_2)$, where $D\subset\mathbb{R}^2$ is a domain, with coordinates (x,y). In the notation of Sec. II, $D = D_z \forall z \in (z_1,z_2)$. We shall also consider a case such that the problem has periodicity in the z direction.

As in Sec. II, we first discuss the existence and uniqueness of the potentials. Then we shall obtain the boundary conditions for the potentials, proving the equivalence between this formulation and the initial Navier–Stokes problem (2). Some of the new boundary conditions are of the integral type, depending on the homology of the domain Ω considered. Because of the particular geometry we have chosen, we obtain, in general, less conditions than in Sec. II.

Proposition 3: Let $\mathbf{v}$ be a continuous field of solenoidal vectors in Ω that verifies (6). If the domain is not periodic in the z direction then it is possible to write

$$\mathbf{v} = \nabla\times(\Psi^{(1)}\hat{\mathbf{e}}_x) + \nabla\times(\Psi^{(2)}\hat{\mathbf{e}}_y). \tag{21}$$

If the domain is periodic in z, $\mathbf{v}$ must verify an additional condition in order to ensure the existence of the potentials $\Psi^{(1)},\Psi^{(2)}$:

$$\int \mathbf{v}\times d\mathbf{l} = 0. \tag{22}$$

In this integral we take (x,y) constant, z extending over a period.

Proof: The existence of a vector potential $\mathbf{B}$ such that $\mathbf{v} = \nabla\times\mathbf{B}$ is guaranteed by (6) as in Proposition 1. If we add a gradient to $\mathbf{B}$ we obtain the same velocity field. Using that freedom we can take the z component of $\mathbf{B}$ to be zero:

$$(\mathbf{B} + \nabla f)\cdot\hat{\mathbf{e}}_z = 0 \Leftrightarrow \frac{\partial f}{\partial z} = -B_z. \qquad (23)$$

If Ω is not periodic in z, then there always exists a solution f. If Ω is z periodic, it is necessary that $\int\mathbf{B}\cdot d\mathbf{l} = 0$ over a period in order to obtain f. But $\int\mathbf{v}\times d\mathbf{l} = -\nabla(\int\mathbf{B}\cdot d\mathbf{l}) = 0$ implies $\int\mathbf{B}\cdot d\mathbf{l} = a$ constant. If λ is the length of the z period, $\int(a/\lambda - B_z)dz = 0$, which guarantees the existence of f such that $\partial f/\partial z = -B_z + a/\lambda$. Taking $\mathbf{v} = \nabla\times[\mathbf{B} - (a/\lambda)\hat{\mathbf{e}}_z + \nabla f]$ we obtain the desired result.

We ourselves suppose that conditions (6) and (22) are a consequence of the boundary conditions of $\mathbf{v}$ on $\partial\Omega$, as in fact occurs in many specific cases. For example, in convection in a rectangular box[5,6] both conditions are trivially satisfied because the homology groups of Ω are trivial. However, in Couette flow it is not possible to use these potentials because (22) is not satisfied. We may now examine the freedom in the choice of potentials (gauge freedom). An elementary calculation shows the following.

Proposition 4: The solution of the homogeneous problem $\nabla\times(\Psi^{(1)}\hat{\mathbf{e}}_x) + \nabla\times(\Psi^{(2)}\hat{\mathbf{e}}_y) = 0$ is given by

$$\Psi^{(1)} = \partial_x f(x,y), \quad \Psi^{(2)} = \partial_y f(x,y), \quad \forall f\in D_z. \qquad (24)$$

So, therefore, we can add to $\Psi^{(1)}$, $\Psi^{(2)}$ the gradient of a horizontal function [depending only of (x,y) coordinates]. If $H_1(D)$ is not trivial, some other terms can be added to the potentials.

We now study the differential equations for $\Psi^{(1)}$, $\Psi^{(2)}$. Taking the curl twice of Navier–Stokes equations we obtain

$$(\partial_t - \nu\Delta)\Delta(\nabla\times\mathbf{B}) = \nabla\times\nabla\times\mathbf{b}. \qquad (25)$$

But the x,y components of $\nabla\times\mathbf{B}$ have a very simple form:

$$v_x = \hat{\mathbf{e}}_x\cdot\nabla\times\mathbf{B} = -\partial_z\Psi^{(2)}, \quad v_y = \hat{\mathbf{e}}_y\cdot\nabla\times\mathbf{B} = \partial_z\Psi^{(1)}. \qquad (26)$$

To obtain equations of even order for the potentials, one takes usually an additional z derivative in (25), obtaining

$$(\partial_t - \nu\Delta)\Delta\partial^2_{zz}\Psi^{(1)} = \hat{\mathbf{e}}_y\cdot\partial_z(\nabla\times\nabla\times\mathbf{b}), \qquad (27a)$$

$$(\partial_t - \nu\Delta)\Delta\partial^2_{zz}\Psi^{(2)} = -\hat{\mathbf{e}}_x\cdot\partial_z(\nabla\times\nabla\times\mathbf{b}), \qquad (27b)$$

and we have eliminated both the pressure and the condition $\nabla\cdot\mathbf{v} = 0$, which is identically satisfied by (21). Since the equations obtained are of order higher than the initial ones, we must add additional boundary conditions so that (2) and (27) become equivalent problems. These additional conditions are given by the following theorem.

Theorem 2: Equations (2) and (27) are equivalent if we add the following boundary conditions to (27):

$$\hat{\mathbf{n}}\cdot\nabla\times[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}] = 0 \quad \text{on } \partial\Omega, \qquad (28a)$$

$$\hat{\mathbf{e}}_z\times\nabla\times[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}] = 0 \quad \text{on } z = z_1,z_2, \qquad (28b)$$

$$\int_c[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}]\cdot d\mathbf{l} = 0, \quad \forall c\in H_1(D). \qquad (28c)$$

Proof: Exactly as in Theorem 1, problem (2) is equivalent to (16). This gives the condition (29c). We have used the property $H_1(\Omega) = H_1(D)$ because of the fact that any closed curve in Ω is homotopically equivalent to a curve in D. Therefore it only remains to prove that (16) is equivalent to (27) and (28). This is a consequence of the following lemma.

Lemma 2: If $\mathbf{A}$ is a solenoidal vector field of vectors in $\Omega = D\times(z_1,z_2)$, then $\mathbf{A} = 0$ in Ω is equivalent to

$$\partial_z\hat{\mathbf{e}}_x\nabla\times\mathbf{A} = 0, \quad \partial_z\hat{\mathbf{e}}_y\nabla\times\mathbf{A} = 0 \quad \text{in } \Omega, \qquad (29a)$$

$$\hat{\mathbf{n}}\cdot\mathbf{A} = 0 \quad \text{on } \partial\Omega, \quad \hat{\mathbf{e}}_z\times\mathbf{A} = 0 \quad \text{on } z = z_1,z_2. \qquad (29b)$$

The direct implication is immediate. To prove the opposite we proceed in several steps. First, we show that $\hat{\mathbf{e}}_z\cdot\nabla\times\mathbf{A} = 0$. From the identity $\nabla\cdot(\partial_z\nabla\times\mathbf{A}) = 0$ we obtain

$$\partial^2_{zz}(\hat{\mathbf{e}}_z\cdot\nabla\times\mathbf{A}) = -\partial_x(\partial_z\hat{\mathbf{e}}_x\cdot\nabla\times\mathbf{A}) - \partial_y(\partial_z\hat{\mathbf{e}}_y\cdot\nabla\times\mathbf{A}) = 0 \qquad (30)$$

as a result of (29a). Condition $\hat{\mathbf{e}}_z\times\mathbf{A} = 0$ in (29b) implies $A_x = A_y = 0$ on $z = z_1,z_2$. Then the tangential derivatives $\partial_y A_x$, $\partial_x A_y$ are also zero, in which case $\hat{\mathbf{e}}_z\cdot\nabla\times\mathbf{A} = 0$ on $z = z_1,z_2$. This last condition and Eq. (30) give rise to $\hat{\mathbf{e}}_z\cdot\nabla\times\mathbf{A} = 0$ in Ω.

Second, we show that $\partial_z\mathbf{A} = 0$ in Ω. By the first step,

$$\partial_z\nabla\times\mathbf{A} = \nabla\times(\partial_z\mathbf{A}) = 0 \Rightarrow \partial_z\mathbf{A} = \nabla f \quad \text{in } \Omega. \qquad (31)$$

In fact, we need some integral conditions, $\int_c\mathbf{A}\cdot d\mathbf{l} = 0$ $\forall c\in H_1(D)$, if $H_1(D)$ is not trivial. But $\int_c\mathbf{A}\cdot d\mathbf{l}$ is identically zero because c always amounts to no more than a combination of the circuits that form $\partial D\times\{z_1\}$, and there $A_x = A_y = 0$. We need to show that f is constant in Ω, but this is a consequence of the Neumann problem satisfied by f:

$$\Delta f = \nabla\cdot\partial_z\mathbf{A} = \partial_z\nabla\cdot\mathbf{A} = 0 \quad \text{in } \Omega, \qquad (32a)$$

$$\frac{df}{dn} = \hat{\mathbf{n}}\cdot\nabla f = \partial_z(\hat{\mathbf{n}}\cdot\mathbf{A}) = 0 \quad \text{on } \partial\Omega. \qquad (32b)$$

The equation $\partial_z(\hat{\mathbf{n}}\cdot\mathbf{A}) = 0$ on $\partial\Omega$ is a consequence of (29b) and $\nabla\cdot\mathbf{A} = 0$.

Third, we show that $\mathbf{A} = 0$ in Ω. By the (29b) condition, $\mathbf{A} = 0$ on $z = z_1, z_2$. By the second step, $\partial_z\mathbf{A} = 0$ in Ω. Integrating with respect to z, considering (x,y) as parameters, and using (29b) we finally obtain the desired result, proving Lemma 2.

We are now able to complete the demonstration of Theorem 2. It suffices to apply Lemma 2 to $\mathbf{A} = \nabla\times[(\partial_t - \nu\Delta)\mathbf{v} + \mathbf{b}]$. Then (29a) changes into (27) and (29b) gives (28a) and (28b).

The integral boundary condition (28c) appears only if the homology group $H_1(D)$ is not trivial. If D is simply connected, only (28a) and (28b) will remain as additional boundary conditions. In order to integrate (27) we need six boundary conditions on $z = z_1, z_2$ and four on ∂D. Equations (28a) and (28b) give three of them on $z = z_1, z_2$ and one on ∂D. The remaining three conditions are the original Navier–Stokes boundary conditions on $\mathbf{v}$. Boundary conditions other than (28) can be used to solve (27). In any case it is necessary to show that these conditions permit us to recover Navier–Stokes equations without additional terms (external forces). The boundary conditions (28), as (15), are stated in terms of velocity field, therefore being independent of the gauge freedom in the potentials. This gauge freedom is usually fixed by the initial conditions.

IV. APPLICATION TO THE PERIODIC COUETTE FLOW

The Couette flow is the motion of a fluid confined between two coaxial cylinders that can rotate independently. We suppose the cylinders to be infinite and the usual hypothesis is to consider the flow periodic in z with a wavelength λ. In cylindrical coordinates the domain Ω will be $(r,\theta,z)\in[R_i,R_0]\times[0,2\pi]\times[0,\lambda]$ and on identifying $\theta=0$ with $\theta=2\pi$, $z=0$ with $z=\lambda$, we obtain a domain that is geometrically equivalent to a toroidal shell (see Fig. 1). The corresponding homology groups are $H_1(\Omega)\simeq\mathbb{R}^2$, $H_1(D_z)\simeq\mathbb{R}$, and $H_2(\Omega)\simeq\mathbb{R}$. That is to say, in Ω there exist two closed curves that are neither homotopic to a point nor homotopic to each other, and that correspond to the closed, nonexact forms $d\theta$, dz. There also exists a closed surface that is nonhomotopic to a point, corresponding to the two-form $d\theta\wedge dz$. Similarly, in D_z there exists a circuit that is nonhomotopic to a point corresponding to $d\theta$.

Since $H_2(\Omega)$ is not trivial the potentials ψ, ϕ might not exist. In that case condition (6) is written as

$$\int_S \mathbf{v}\cdot d\mathbf{S}=\int_{r=R_0} v_r\,dS=0, \tag{33}$$

which is automatically satisfied since radial velocity cancels out at the cylinder walls. In fact, the boundary conditions for $\mathbf{v}$ are $\mathbf{v}=R_i\Omega_i\hat{\mathbf{e}}_\theta$ on $r=R_i$, $\mathbf{v}=R_0\Omega_0\hat{\mathbf{e}}_\theta$ or $r=R_0$, which, if written in terms of potentials, give

$$\begin{aligned}&(1/r)\psi_\theta+\phi_{rz}=0,\quad -\psi_r+(1/r)\phi_{\theta z}=R_j\Omega_j,\\&\Delta_h\phi=0\quad\text{on}\quad r=R_i,R_0.\end{aligned} \tag{34}$$

For the Couette flow there is no problem regarding existence, uniqueness, and regularity of $\mathbf{v}$. In fact (see Temam[20] p. 303), if the initial conditions $\mathbf{v}(t=0)$ are C^∞ then $\mathbf{v}$ exists, is unique, and $\mathbf{v}\in C^\infty\,[\overline{\Omega}\times(0,\infty)]$.

The additional boundary conditions (15) can be calculated easily and give

$$\Delta^2\phi_\theta=rD\Delta\psi_z\quad\text{on}\quad r=R_i,R_0, \tag{35a}$$

$$\int_0^{2\pi}[(\partial_t-\nu\Delta)\mathbf{v}+\mathbf{b}]\cdot\hat{\mathbf{e}}_\theta\,d\theta=-\int_0^{2\pi}[(\partial_t-\nu\tilde{\Delta})D\psi-\hat{\mathbf{e}}_\theta\cdot\mathbf{b}]d\theta=0, \tag{35b}$$

$$\int_0^{\lambda}[(\partial_t-\nu\Delta)\mathbf{v}+\mathbf{b}]\cdot\hat{\mathbf{e}}_\theta\,dz=-\int_0^{\lambda}[(\partial_t-\nu\tilde{\Delta}_h)\Delta_h\phi-\hat{\mathbf{e}}_z\cdot\mathbf{b}]dz=0, \tag{35c}$$

$$\int_0^{2\pi}\nabla\times[(\partial_t-\nu\Delta)\mathbf{v}+\mathbf{b}]\cdot\hat{\mathbf{e}}_\theta\,d\theta=+\int_0^{2\pi}[(\partial_t-\nu\tilde{\Delta})\tilde{\Delta}\,D\phi+\hat{\mathbf{e}}_\theta\cdot\nabla\times\mathbf{b}]d\theta=0, \tag{35d}$$

where we have used the symbols $D=\partial_r$, $D_+=D+1/r$, and $\tilde{\Delta}=DD_++\partial_{zz}^2$. Integrals in (35) are calculated on $r=R_i$, R_0. In fact, as a consequence of (34), time derivatives disappear in (35b) and (35c). We keep them in order to simplify some calculations in Appendix B. Equations (35b) and (35c) ensure that pressure is periodic in θ and z directions. If Eq. (35d) is not imposed, an easy calculation following Lemma 1 gives

$$(\partial_t-\nu\Delta)\mathbf{v}+\mathbf{b}+C\ln(r/R_i)\hat{\mathbf{e}}_z=0. \tag{36}$$

Those are the Navier–Stokes equations with an additional external force $C\ln(r/R_i)\hat{\mathbf{e}}_z$. This force field is variable in space, and it is not a gradient, giving as a consequence a different fluid motion than the original one. The constant C that appears in (36) depends on the initial and boundary conditions used to solve (14). If Eq. (35a) is not imposed, we obtain analogously

$$(\partial_t-\nu\Delta)\mathbf{v}+\mathbf{b}+\chi\hat{\mathbf{e}}_z=0, \tag{37}$$

where χ is a harmonic horizontal function $\Delta_h\chi=0$. If χ depends explicitly on r or θ then it is not a gradient, giving rise to a fluid motion different than the original Navier–Stokes problem.

Equations (34) and (35) are the boundary conditions for (14). As we have three integrals plus time derivatives in the boundary conditions, we might believe that this formulation is not really suitable for solving the Couette problem. We shall see below that all these drawbacks disappear, bringing about a reduction in the order of Eqs. (14). The key idea is that in a Fourier development of ψ, ϕ in θ, z directions, Eqs. (35b)–(35d) give ordinary boundary conditions above the zero mode. And the zero mode of Eqs. (14a) and (14b) has the form of a total derivative. Combining these facts we can reduce the order of the zero mode for ψ, ϕ and the integral boundary conditions disappear. The details can be seen in Appendix B. We extract the zero mode using projection operators P_θ, P_z and writing ψ, ϕ as

$$\psi=P_\theta\psi+\tilde{\psi},\quad\phi=P_\theta\phi+P_z(1-P_\theta)\phi+\tilde{\phi}, \tag{38}$$

where $\tilde{\psi}$, $\tilde{\phi}$ are zero-mode-free,

$$P_\theta\tilde{\psi}=0,\quad P_\theta\tilde{\phi}=P_z\tilde{\phi}=0; \tag{39}$$

$P_\theta\psi$, $P_\theta\phi$, and $P_z(1-P_\theta)\phi$ always appear in the combinations

$$\begin{aligned}&f=-DP_\theta\,\psi,\quad g=-DP_\theta\phi,\\&h=-\Delta_hP_z(1-P_\theta)\phi;\end{aligned} \tag{40}$$

f, g, and h are the averages of the potentials ψ, ϕ with respect to z, θ and therefore $f(r,z)$, $g(r,z)$, and $h(r,\theta)$; also, $P_\theta h=0$.

In terms of those new variables we obtain

$$(\partial_t-\nu\tilde{\Delta})f=-P_\theta\hat{\mathbf{e}}_\theta\cdot\mathbf{b}, \tag{41a}$$

$$(\partial_t-\nu\Delta_h)h=-P_z(1-P_\theta)\hat{\mathbf{e}}_z\cdot\mathbf{b}, \tag{41b}$$

$$(\partial_t-\nu\tilde{\Delta})\tilde{\Delta}g=P_\theta\hat{\mathbf{e}}_\theta\cdot\nabla\times\mathbf{b}, \tag{41c}$$

$$(\partial_t-\nu\Delta)\Delta_h\tilde{\psi}=(1-P_\theta)\hat{\mathbf{e}}_z\cdot\nabla\times\mathbf{b}, \tag{41d}$$

$$(\partial_t-\nu\Delta)\Delta\Delta_h\tilde{\phi}=-(1-P_\theta)(1-P_z)\hat{\mathbf{e}}_z\cdot\nabla\times\nabla\times\mathbf{b}, \tag{41e}$$

which corresponds to the following decomposition for the velocity:

$$\mathbf{v}=f\hat{\mathbf{e}}_\theta+h\hat{\mathbf{e}}_z+\nabla\times(g\hat{\mathbf{e}}_\theta+\tilde{\psi}\hat{\mathbf{e}}_z)+\nabla\times\nabla\times(\tilde{\phi}\hat{\mathbf{e}}_z). \tag{42}$$

The boundary conditions for these potentials are (34) and (35a), written in terms of $(f,g,h,\tilde{\psi},\tilde{\phi})$, to which we must add those originating from the choice of gauge freedom. We write (12a) in terms of the new potentials,

$$\Delta_h \phi = 0 \Rightarrow \Delta_h \tilde{\phi} = D_+ g + h, \quad (43a)$$

$$\int_c \frac{d\phi_z}{dn} dl = -2\pi r g_z = 0 \quad \text{on} \quad r = R_i, R_0. \quad (43b)$$

By taking $g = 0$ in R_i, R_0 automatically satisfies the second condition and since now there are no restrictions regarding $\tilde{\phi}$, we can take $\tilde{\phi} = 0$ on R_i, R_0. By using the conditions $g = \tilde{\phi} = 0$ on R_i, R_0 and writing the boundary conditions (34) and (35a) in terms of the new potentials, we have

$$f = R_j \Omega_j, \quad h = 0, \quad g = g_r = 0, \quad (44a)$$

$$\tilde{\psi}_r = 0, \quad \tilde{\phi} = \Delta_h \tilde{\phi} = 0, \quad (44b)$$

$$\tilde{\psi}_\theta + r\tilde{\phi}_{rz} = 0, \quad \Delta\Delta_h \tilde{\phi}_\theta = rD\Delta_h \tilde{\psi}_z \quad (44c)$$

on $r = R_j$, $j = i, 0$. Equations (41) and (44) are equivalent to the Navier–Stokes equations for the Couette flow. They have been written in terms of the potentials of velocity and contain neither the pressure nor the condition $\nabla \cdot \mathbf{v} = 0$. Moreover, the boundary conditions (44) are reasonably simple as they contain neither derivatives regarding t nor integral conditions.

As an immediate application we are able to obtain the equations for the Taylor vortex flow that correspond to the axisymmetric Couette problem. Once the dependence on θ is eliminated, we can put $h = \tilde{\psi} = \tilde{\phi} = 0$ (because $\partial_\theta F = 0$ and $P_\theta F = 0$ implies $F = 0$) and we are left with

$$(\partial_t - \nu\tilde{\Delta})f = -\hat{\mathbf{e}}_\theta \cdot \mathbf{b}, \quad (45a)$$

$$(\partial_t - \nu\tilde{\Delta})\tilde{\Delta} g = \hat{\mathbf{e}}_\theta \cdot \nabla \times \mathbf{b}, \quad (45b)$$

with boundary conditions $f = R_j$, Ω_j, $g = g_r = 0$ on $r = R_j$, $j = i, 0$ and velocity given by $\mathbf{v} = f\hat{\mathbf{e}}_\theta + \nabla \times (g\hat{\mathbf{e}}_\theta)$. That is to say, f is the azimuthal component of velocity and g is the velocity potential, which gives us the current lines in the plane (r,z). These equations coincide (up to notation) with those given by other authors.[21] We note that (45) are the θ components of the Navier–Stokes equations and their curl.

V. CONCLUSION

The techniques developed in this work can be applied to any incompressible fluid flow problem. In particular, the formulation obtained [(42) and (45)] for the Couette problem in terms of potentials is completely general; it is not submitted to the restrictions of other formulations found in the literature, which are valid only in the axisymmetrical problem[21] or in the narrow gap approximation.[4] It is worth mentioning that the set of equations [(41) and (44)] is very well suited for standard numerical integration techniques, as they are a system of generalized quasilinear parabolic equations.

We have studied in this paper two potential methods well suited to solve the Navier–Stokes equations for incompressible fluids. There are two major advantages in these methods. First of all, they eliminate the pressure and the incompressibility condition $\nabla \cdot \mathbf{v} = 0$, giving two scalar equations for the potentials. Second, a necessary and sufficient set of boundary conditions has been derived that guarantees the equivalence with the original Navier–Stokes problem; this is the main contribution of this paper. However, the boundary conditions obtained couple both potentials and, depending on the geometry of the fluid domain, they may become integral boundary conditions. We think these integral boundary conditions can be circumvented in most cases, as in the Couette flow discussed in Sec. IV, by using similar tricks. In numerical applications, by using spectral methods, these integral conditions give algebraic linear equations between the spectral coefficients of the potentials.

Recent developments on spectral methods together with the use of supercomputers make it possible to calculate flows in complex geometries[22] as well as to simulate three-dimensional flows.[6] Potential methods have been used by several authors,[3,5,6,18] but the advantages and disadvantages of the methods discussed in this paper cannot be made apparent unless more numerical work and a detailed comparative numerical analysis are undertaken.

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APPENDIX A: OTHER CHOICES FOR e

We shall now give the $\mathbf{e}$ components of velocity and its curl, as well as the curl and double-curl of the Navier–Stokes equations for different choices of $\mathbf{e}$.

Case 1. $\mathbf{e} = (0,0,1)$:

$$\mathbf{e} \cdot \mathbf{v} = -\Delta_h \phi, \quad \mathbf{e} \cdot \nabla \times \mathbf{v} = -\Delta_h \psi, \quad (A1a)$$

$$(\partial_t - \nu\Delta)\Delta_h \psi = \mathbf{e} \cdot \nabla \times \mathbf{b}, \quad (A1b)$$

$$(\partial_t - \nu\Delta)\Delta\Delta_h \phi = -\mathbf{e} \cdot \nabla \times \nabla \times \mathbf{b}. \quad (A1c)$$

Case 2. $\mathbf{e} = \mathbf{r} = (x, y, z)$:

$$\mathbf{e} \cdot \mathbf{v} = -\Delta_S \phi, \quad \mathbf{e} \cdot \nabla \times \mathbf{v} = -\Delta_S \psi, \quad (A2a)$$

$$(\partial_t - \nu\Delta)\Delta_S \psi = \mathbf{e} \cdot \nabla \times \mathbf{b}, \quad (A2b)$$

$$(\partial_t - \nu\Delta)\Delta\Delta_S \phi = -\mathbf{e} \cdot \nabla \times \nabla \times \mathbf{b}, \quad (A2c)$$

where

$$\Delta_S = (1/\sin\theta)\partial_\theta(\sin\theta\partial_\theta) + (1/\sin^2\theta)\partial^2_{\varphi\varphi} \quad (A3)$$

is the angular part of the Laplace operator in spherical coordinates.

Case 3. $\mathbf{e} = (x, y, 0)$:

$$\mathbf{e} \cdot \mathbf{v} = -r^2\Delta_C \phi, \quad \mathbf{e} \cdot \nabla \times \mathbf{v} = -r^2\Delta_C \psi + 2\phi_{\theta z} \quad (A4)$$

and so the association of ψ with $\mathbf{e} \cdot \nabla \times \mathbf{v}$ is lost;

$$(\partial_t - \nu\overline{\Delta})\Delta_C \psi - \frac{2}{r^2}(\partial_t - 2\nu\Delta)\phi_{\theta z} - \frac{2\nu}{r^3}\psi_{r\theta\theta} = \frac{1}{r^2}\mathbf{e} \cdot \nabla \times \mathbf{b}, \quad (A5a)$$

$$(\partial_t - \nu\overline{\Delta})\left(\overline{\Delta}\Delta_C \phi + \frac{2}{r^3}\phi_{r\theta\theta}\right) - \frac{2}{r^2}(\partial_t - 2\nu\Delta)\phi_{\theta z} - \frac{2\nu}{r^3}\left(D\Delta\phi_{\theta\theta} - \frac{2}{r}\phi_{\theta\theta zz}\right) = -\frac{1}{r^2}\mathbf{e} \cdot \nabla \times \nabla \times \mathbf{b}, \quad (A5b)$$

where we have used cylindrical coordinates and $\Delta_C = (1/r^2)\partial^2_{\theta\theta} + \partial^2_{zz}$ is the cylindrical part of the Laplacian operator. These equations may seem more complicated than those of the two preceding cases, because their linear part (the left-hand sides) couples the potentials ψ and ϕ. The reader can readily grasp that no such difficulty exists either in the axisymmetric case or in the narrow gap approximation for the Couette flow. Nevertheless, Eqs. (A5) have several good properties. They are of second and fourth order in r, respectively, while in case 1 Eqs. (A1b) and (A1c) are of the fourth and sixth order in r. And the boundary conditions on surfaces $r=$ const are very simple: $\psi = \phi = D\phi = 0$. Hence in this case we do not need additional boundary conditions.

Finally, we can calculate the potentials ψ, ϕ corresponding to the basic Couette flow: $\mathbf{v} = (Ar + B/r)\hat{\mathbf{e}}_\theta$, where A and B are constants such that $v_\theta = R_i\Omega_i$ on $r = R_i$, $v_\theta = R_0\Omega_0$ on $r = R_0$. By choosing $\mathbf{e} = (x, y, 0)$ we obtain $\phi = 0$, $\psi = (A + B/r^2)z$. In this case $\mathbf{v}$ is independent of z but ψ depends on it and is not periodic, and hence no solution exists. In contrast, by taking $\mathbf{e} = (0,0,1)$ we obtain $\phi = 0$, $\psi = -Ar^2/2 - B\log(r)$, which does satisfy all requirements. The nonperiodic character of the potentials ψ, ϕ can be circumvented in several ways. One possibility is to work directly with nonperiodic ψ, ϕ, but guaranteeing that $\mathbf{v}$ is periodic. Another method is to take $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, where $\mathbf{v}_0$ is known and includes the nonperiodic part of ψ, ϕ. In the Couette flow, for example, we can take $\mathbf{v}_0 = (Ar + B/r)\hat{\mathbf{e}}_\theta$ and then put $\mathbf{v}' = \nabla\times(\psi\mathbf{e}) + \nabla\times\nabla\times(\phi\mathbf{e})$ with ψ, ϕ periodic.

APPENDIX B: REDUCTION OF THE ORDER

We shall now obtain the reduction in the order of Eqs. (14) for the Couette flow. Let us first study the potential ψ. We shall introduce the following average operator:

$$P_\theta F = \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta,z,t)d\theta. \quad \text{(B1)}$$

Obvious properties of P_θ are

$$\partial_\theta F = 0 \Rightarrow P_\theta F = F, \quad \partial_\theta P_\theta F = P_\theta \partial_\theta F = 0, \quad \text{(B2)}$$

where we have used the periodicity of F with respect to θ. Let us now break ψ down into the form

$$\psi = P_\theta \psi + (1 - P_\theta)\psi = P_\theta \psi + \tilde{\psi}. \quad \text{(B3)}$$

From Eq. (14a) and through application of P_θ and $(1 - P_\theta)$ to it we obtain equations for $P_\theta \psi$ and $\tilde{\psi}$. By applying P_θ to (14a) we obtain

$$D_+[(\partial_t - \nu\tilde{\Delta})DP_\theta \psi - P_\theta(\hat{\mathbf{e}}_\theta\cdot\mathbf{b})] = 0. \quad \text{(B4)}$$

Equation (35b) can now be written as

$$(\partial_t - \nu\tilde{\Delta})DP_\theta \psi - P_\theta(\hat{\mathbf{e}}_\theta\cdot\mathbf{b}) = 0 \quad \text{on} \quad r = R_i. \quad \text{(B5)}$$

By integrating (B4) with respect to r and using (B5) we obtain

$$(\partial_t - \nu\tilde{\Delta})DP_\theta \psi = P_\theta(\hat{\mathbf{e}}_\theta\cdot\mathbf{b}). \quad \text{(B6)}$$

On the other hand,

$$\nabla\times\psi\hat{\mathbf{e}}_z = -\hat{\mathbf{e}}_\theta DP_\theta\psi + \nabla\times\tilde{\psi}\hat{\mathbf{e}}_z. \quad \text{(B7)}$$

Now, as $P_\theta \psi$ always appears in the combination $DP_\theta\psi$, by introducing $f = -DP_\theta \psi$ we finally obtain

$$(\partial_t - \nu\tilde{\Delta})f = -P_\theta(\hat{\mathbf{e}}_\theta\cdot\mathbf{b}), \quad \text{(B8)}$$

with which we have reduced the order of the equation by 2 for $P_\theta \psi$. For this we have used the integral condition (35b) and the fact that $P_\theta \psi$ always appears in $\mathbf{v}$ derived with respect to r. We see that (B8) is nothing but the θ component of the Navier–Stokes equations projected by P_θ, and pressure disappears.

The equation for $\tilde{\psi}$ does not present any difficulty. It gives

$$(\partial_t - \nu\Delta)\Delta_h\tilde{\psi} = (1 - P_\theta)\hat{\mathbf{e}}_z\cdot\nabla\times\mathbf{b}. \quad \text{(B9)}$$

The reduction in the order of ϕ is more complicated, however. As well as P_θ we also need the operator

$$P_z F = \frac{1}{\lambda}\int_0^\lambda F(r,\theta,z,t)dz, \quad \text{(B10)}$$

which averages out F with respect to z. As before,

$$\partial_z F = 0 \Rightarrow P_z F = F, \quad \partial_z P_z F = P_z \partial_z F = 0, \quad \text{(B11)}$$

where we have used the periodicity of F with respect to z. The decomposition of ϕ will therefore be

$$\phi = P_\theta \phi + P_z(1 - P_\theta)\phi + (1 - P_\theta)(1 - P_z)\phi$$
$$= P_\theta \phi + P_z(1 - P_\theta)\phi + \tilde{\phi}, \quad \text{(B12)}$$

where we have primarily extracted the average with respect to θ, and then subsequently the average of the result with respect to z. In order to obtain the corresponding equations for the three terms of (B12) we proceed as before. A simple calculation gives

$$D_+[(\partial_t - \nu\tilde{\Delta})\tilde{\Delta}\, DP_\theta \phi + P_\theta(\hat{\mathbf{e}}_\theta\cdot\nabla\times\mathbf{b})] = 0, \quad \text{(B13a)}$$

$$(\partial_t - \nu\tilde{\Delta})\tilde{\Delta}\, DP_\theta \phi + P_\theta(\hat{\mathbf{e}}_\theta\cdot\nabla\times\mathbf{b}) = 0 \quad \text{on} \quad r = R_i, \quad \text{(B13b)}$$

$$\nabla\times\nabla\times(\phi\hat{\mathbf{e}}_z) = -\Delta_h P_z(1 - P_\theta)\phi\hat{\mathbf{e}}_z - \nabla\times(DP_\theta \phi\hat{\mathbf{e}}_\theta) + \nabla\times\nabla\times(\tilde{\phi}\hat{\mathbf{e}}_z). \quad \text{(B13c)}$$

Then $P_\theta \phi$ always appears in $\mathbf{v}$ in the form $g = -DP_\theta \phi$; by integrating (B13a) and using (B13b) we finally obtain

$$(\partial_t - \nu\tilde{\Delta})\tilde{\Delta}g = P_\theta(\hat{\mathbf{e}}_\theta\cdot\nabla\times\mathbf{b}). \quad \text{(B14)}$$

Thanks to the integral condition (35d) we have been able to reduce the order by 2. We may observe that (B14) is no more than the component θ of the curl of the Navier–Stokes equations projected by P_θ. By applying $P_z(1 - P_\theta)$ to (14b) and writing (35c) in terms of P_z we obtain

$$\Delta_h[(\partial_t - \nu\Delta_h)\Delta_h P_z(1 - P_\theta)\phi - P_z(1 - P_\theta)\hat{\mathbf{e}}_z\cdot\mathbf{b}] = 0. \quad \text{(B15a)}$$

$$(\partial_t - \nu\Delta_h)\Delta_h P_z \phi - P_z(\hat{\mathbf{e}}_z\cdot\mathbf{b}) = 0 \quad \text{on} \quad r = R_i, R_0. \quad \text{(B15b)}$$

By applying $(1 - P_\theta)$ to (B15b) we can integrate (B15a) to obtain

$$(\partial_t - \nu\Delta_h)h = -P_z(1 - P_\theta)\hat{\mathbf{e}}_z\cdot\mathbf{b}, \quad \text{(B16)}$$

where we have called $h = -\Delta_h P_z(1 - P_\theta)\phi$. The integral condition (35c) has allowed us to reduce the order of the equation for $P_z(1-P_\theta)\phi$ by 4. We note that (B16) is the z component of the Navier–Stokes equations applying $P_z(1-P_\theta)$, and the pressure disappears. Finally, the equation for $\tilde{\phi}$ reads

$$(\partial_t - \nu\Delta)\Delta\Delta_h \tilde{\phi} = -(1-P_\theta)(1-P_z)\hat{\mathbf{e}}_z\cdot\nabla\times\nabla\times\mathbf{b}. \quad \text{(B17)}$$

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