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Nonnormal Effects in the Taylor–Couette Problem*

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Abstract. This work is devoted to the study of transient growth of perturbations in the Taylor–Couette problem due to linear nonnormal mechanisms. The study is carried out for a particular small gap case and is mostly focused on the linearly stable regime of counter-rotation. The exploration covers a wide range of inner and outer angular speeds as well as axial and azimuthal modes. Significant transient growth is found in the regime of stable counter-rotation. The numerical results are in agreement with former analyses based on energy methods and other independent numerical studies. The optimal energy transient growth factor appears to be consistent with experimental observations. This study might shed some light on the subcritical transition to turbulence which is found experimentally in Taylor–Couette flow when the cylinders rotate in opposite directions.

1. Introduction

Taylor–Couette flow of a viscous fluid confined between independently rotating coaxial cylinders has been one of the most studied problems of fluid dynamics in the last 80 years. Starting with the celebrated work of Taylor (1923), the Taylor–Couette problem has been an experimental, theoretical and numerical benchmark problem for bifurcation theory and hydrodynamic stability. This flow may become turbulent by means of many different mechanisms which usually involve successive steady or unsteady linear instabilities. The flow may exhibit many different steady, time periodic or almost periodic patterns before an eventual transition to chaotic regimes (Andereck *et al.*, 1986; DiPrima and Swinney, 1981). We refer the reader to the standard monographs by Chossat and Iooss (1991) or Tagg (1994) for details. Below the critical values predicted by linear stability theory, azimuthal Couette flow is stable with respect to infinitesimal perturbations. Nevertheless, experiments formerly carried out by Coles (1965) and Van Atta (1966), and later by Hegseth *et al.* (1989), reported striking new phenomena of sudden transition to *spiral turbulence* in the region where the linear theory predicted stability of the basic azimuthal Couette flow. This kind of instability, which Coles termed *catastrophic transition*, cannot be explained by means of eigenvalue analysis of the linearized Navier–Stokes operator. Instead, this subcritical transition may be associated with the considerable amplification or transient growth that even very small amplitude perturbations may suffer due to the nonnormality of the linearized operator, i.e., nonorthogonality of its eigenvectors (Kato, 1976). It has long been known that nonnormality of linearized operators of pipe (Boberg and Brosa, 1988), plane Poiseuille

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(Gustavsson, 1991) or plane Couette flows (Butler and Farrell, 1992) is responsible for the considerable nonmodal linear growth of small perturbations. Plane Couette or pipe Poiseuille flows are linearly stable for all Reynolds numbers (Romanov, 1973; Drazin and Reid, 1981) although they actually become turbulent due to finite amplitude perturbations which are transiently amplified by nonnormal mechanisms. The question regarding the role of nonnormality in subcritical transition of shear flows has generated many controversies during the past decade (Trefethen *et al.*, 1993; Waleffe, 1995), Reddy and Henningson (1993) being the first attempt at clarification. A comprehensive theoretical study of nonmodal analysis for this type of flows can be found in the recently published monograph by Schmid and Henningson (2001).

In Hristova *et al.* (2001) a first nonmodal analysis of the linearized Taylor–Couette problem has been provided for axisymmetric perturbations with fixed axial periodicity. Although the nonnormality of the Taylor–Couette problem was first pointed out in Gebhardt and Grossmann (1993), this feature has been studied in *et al.* (2001) by means of the computation of the pseudospectra of the linear operator (Trefethen, 1999). The exploration in *et al.* (2001) was carried out for different values of the radius ratio of the cylinders and for a fixed angular speed ratio so that the average angular speed eliminates the Coriolis effect in the narrow-gap limit. Their purpose was to recover the plane Couette behaviour as a narrow gap limit of the Taylor–Couette problem. One of the motivations of this research is the remarkable similarities of the spiral turbulent patterns between plane Couette and narrow-gap Taylor–Couette flow which have been recently reported by Prigent and Dauchot (2001).

The experiments of Coles and Van Atta were carried out with a narrow-gap apparatus and subcritical transition to turbulence was found in the regime of counter-rotation or when the inner cylinder was at rest. The purpose of this work is to examine the transient energy growth of perturbations based on the linear nonmodal analysis of the azimuthal Couette flow under those circumstances. The author *does not* claim that this mechanism is the only one responsible for the eventual transition to turbulence; nonlinear effects are also crucial for that transition.

The paper is structured as follows. In Section 2 we formulate the stability problem and we define the quantities which measure the transient growth of the perturbations. In Section 3 we provide a comprehensive exploration of the optimal transient growth in the counter-rotation regime for different azimuthal and axial modes, and we compare our numerical results with the experimental data available, former theoretical works based on energy methods and with a former nonmodal linear growth analysis.

2. Mathematical Formulation: Linear Stability and Energy Norm

We consider an incompressible fluid of kinematic viscosity ν and density ρ which is contained between two concentric rotating cylinders whose inner and outer radii and angular velocities are r_{i}^*, r_{o}^* and Ω_{i}, Ω_{o}, respectively. Henceforth, all variables will be rendered dimensionless using $d = r_{\mathrm{o}}^* - r_{\mathrm{i}}^*$, d^2/ν, ν^2/d^2 as units for space, time and the reduced pressure (p/ρ), respectively. The independent dimensionless parameters appearing in this problem are the radius ratio $\eta = r_{\mathrm{i}}^*/r_{\mathrm{o}}^*$ which fixes the geometry of the annulus, and the Couette flow Reynolds numbers $Ri = \mathrm{d}r_{\mathrm{i}}\Omega_{\mathrm{i}}/\nu$ and $Ro = \mathrm{d}r_{\mathrm{o}}\Omega_{\mathrm{o}}/\nu$ of the rotating cylinders. The Navier–Stokes equation and the incompressibility condition for this scaling take the form

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \Delta \mathbf{v}, \qquad \nabla\cdot\mathbf{v} = 0. \tag{1}$$

Let $\mathbf{v} = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z = (v_r, v_\theta, v_z)$ be the velocity vector $\mathbf{v}$ in cylindrical coordinates (r, θ, z). The basic azimuthal Couette flow $\mathbf{v}^{\mathrm{B}} = (v_r^{\mathrm{B}}, v_\theta^{\mathrm{B}}, v_z^{\mathrm{B}})$ is obtained by assuming independence with respect to t, θ and z:

$$v_r^{\mathrm{B}} = 0, \qquad v_\theta^{\mathrm{B}} = Ar + \frac{B}{r}, \qquad v_z^{\mathrm{B}} = 0 \qquad (r_{\mathrm{i}} \le r \le r_{\mathrm{o}}), \tag{2}$$

where $A = (Ro - \eta Ri)/(1+\eta)$, $B = \eta(Ri - \eta Ro)/(1-\eta)(1-\eta^2)$, $r_{\mathrm{i}} = \eta/(1-\eta)$ and $r_{\mathrm{o}} = 1/(1-\eta)$.

For our analysis, the basic flow is perturbed by a small disturbance which is assumed to be periodic in the azimuthal and axial coordinates:

$$\mathbf{v}(r,\theta,z,t) = \mathbf{v}^{\mathrm{B}} + \mathbf{u}(r)\,\mathrm{e}^{\mathrm{i}(n\theta+kz)+\lambda t}, \tag{3}$$

$$p(r,\theta,z,t) = p^{\mathrm{B}} + q(r)\,\mathrm{e}^{\mathrm{i}(n\theta+kz)+\lambda t}, \tag{4}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. In addition, the perturbation of the velocity field, $\mathbf{u} = (u_r, u_\theta, u_z)$, must vanish at the radial boundaries

$$\mathbf{u}(r_\mathrm{i}) = \mathbf{u}(r_\mathrm{o}) = \mathbf{0}, \tag{5}$$

and satisfy the solenoidal condition

$$\nabla \cdot [\mathrm{e}^{\mathrm{i}(n\theta + kz)}\mathbf{u}(r)] = 0. \tag{6}$$

By introducing the perturbed fields (3) and (4) in the Navier–Stokes equations (1) and neglecting nonlinear terms, we obtain the solenoidal eigenvalue problem for the (n, k) azimuthal–axial mode of the perturbation

$$\lambda u_r = -\mathrm{D}q + \left[\mathrm{D}_+\mathrm{D} - \frac{n^2+1}{r^2} - k^2 - \frac{\mathrm{i}n}{r}v_\theta^\mathrm{B}\right] u_r + \left[\frac{2}{r}v_\theta^\mathrm{B} - \frac{2\mathrm{i}n}{r^2}\right] u_\theta\ , \tag{7}$$

$$\lambda u_\theta = -\frac{\mathrm{i}n}{r}q + \left[\mathrm{D}_+\mathrm{D} - \frac{n^2+1}{r^2} - k^2 - \frac{\mathrm{i}n}{r}v_\theta^\mathrm{B}\right] u_\theta + \left[\frac{2\mathrm{i}n}{r^2} - \left(\mathrm{D}_+ v_\theta^\mathrm{B}\right)\right] u_r\ , \tag{8}$$

$$\lambda u_z = -\mathrm{i}kq + \left[\mathrm{D}_+\mathrm{D} - \frac{n^2}{r^2} - k^2 - \frac{\mathrm{i}n}{r}v_\theta^\mathrm{B}\right] u_z\ , \tag{9}$$

$$\mathrm{D}_+ u_r = -\frac{\mathrm{i}n}{r}u_\theta - \mathrm{i}ku_z\ , \tag{10}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}r$ and $\mathrm{D}_+ = \mathrm{D} + 1/r$.

For a fixed (n, k)-mode, we discretize the boundary value problem (5)–(10) by a solenoidal Petrov–Galerkin spectral method described in Canuto *et al.* (1988), p. 228, whose accuracy was confirmed in Meseguer and Marques (2000) for the stability analysis of the spiral Couette flow. The discretization scheme leads to an eigenvalue problem for the amplitudes $\mathbf{a} = (a_0, \ldots, a_\mathrm{M})^\mathrm{T}$ of the spectral representation of the velocity field:

$$\mathbb{L}(Ri, Ro, \eta, n, k)\mathbf{a} = \lambda\mathbf{a}, \tag{11}$$

where the matrix $\mathbb{L}$ implicitly depends on the set of parameters of the boundary value problem. The linear stability problem is then reduced to the computation of the spectrum of $\mathbb{L}$ for each pair of (n, k) azimuthal–axial modes. If, for a fixed set of values Ri, Ro and η, the (n, k)-spectra always lie in the left-hand side of the complex plane, then the basic flow will be stable with respect to infinitesimal perturbations. On the other hand, if one of the eigenvalues has positive real part, then the basic Couette flow will be linearly unstable.

We focus our attention in the transient evolution of perturbations in the regime of linear stability, following the same methodology used in Schmid and Henningson (1994) for the study of nonnormal transient growth in Hagen–Poiseuille flow. For a given (n, k) azimuthal–axial mode, consider the linear subspace S_N spanned by the eigenvectors of the N rightmost eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the spectrum of $\mathbb{L}$,

$$S_N = \langle \tilde{\mathbf{q}}_1, \tilde{\mathbf{q}}_2, \ldots, \tilde{\mathbf{q}}_N \rangle. \tag{12}$$

Any perturbation $\mathbf{q} \in S_N$ can be expressed as a linear combination of the eigenvectors $\tilde{\mathbf{q}}_i$,

$$\mathbf{q} = \sum_{n=1}^{N} \kappa_n \tilde{\mathbf{q}}_n = (\kappa_1, \kappa_2, \ldots, \kappa_N)^\mathrm{T}, \tag{13}$$

and its time evolution is dictated by the diagonal system

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t} = \Lambda\boldsymbol{\kappa}, \tag{14}$$

where we have assumed that we have distinct eigenvalues and eigenvectors and where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_N)^\mathrm{T}$ and $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. We define the *energy norm* of the perturbation $\mathbf{q}$ by means of the inner product

$$\varepsilon(\mathbf{q}) = (\mathbf{q}, \mathbf{q})_{\mathrm{E}} = \frac{1}{2} \int_{r_{\mathrm{i}}}^{r_{\mathrm{o}}} \mathbf{q}^{*} \cdot \mathbf{q} r \, \mathrm{d}r, \tag{15}$$

where $*$ stands for complex conjugation. We consider the matrix of inner products between the eigenvectors

$$\mathbb{M}_{ij} = (\tilde{\mathbf{q}}_i, \tilde{\mathbf{q}}_j)_{\mathrm{E}}. \tag{16}$$

This matrix is positive definite and it admits a decomposition of the form $\mathbb{M} = \mathbb{F}^{\dagger}\mathbb{F}$, where $\dagger$ stands for the complex conjugate transpose. This decomposition can be accomplished by means of the standard QR factorization. The energy norm of the perturbation $\mathbf{q}$ in (15) can be expressed in the standard 2-norm in S_N by means of the components $\mathbb{F}$ and $\mathbb{F}^{\dagger}$:

$$\varepsilon(\mathbf{q}) = \boldsymbol{\kappa}^{\dagger}\mathbb{M}\boldsymbol{\kappa} = (\mathbb{F}\boldsymbol{\kappa}, \mathbb{F}\boldsymbol{\kappa})_2 = (\boldsymbol{\kappa}, \boldsymbol{\kappa})_{\mathrm{E}} = ||\boldsymbol{\kappa}||_{\mathrm{E}}^2 = ||\mathbb{F}\boldsymbol{\kappa}||_2^2.$$

We are interested in the measurement of the energy growth of an initial condition $\boldsymbol{\kappa}_0$ as a function of time. Following Boberg and Brosa (1988), we define the *energy amplification factor*, $g(t)$, as the ratio between the energy norm of the perturbation at time t and its initial norm,

$$g(t) = \frac{||\boldsymbol{\kappa}(t)||_{\mathrm{E}}^2}{||\boldsymbol{\kappa}_0||_{\mathrm{E}}^2} = \frac{||\mathrm{e}^{\Lambda t}\boldsymbol{\kappa}_0||_{\mathrm{E}}^2}{||\boldsymbol{\kappa}_0||_{\mathrm{E}}^2}. \tag{17}$$

For a fixed time t, we want to maximize $g(t)$ in (17) over the set of all possible initial conditions κ_0. Maximization of the ratio appearing in (17) leads to the quantity $G(t)$, the *optimal energy amplification factor*:

$$G(t) = \max_{||\boldsymbol{\kappa}_0|| \neq 0} g(t) = \max_{||\boldsymbol{\kappa}_0|| \neq 0} \frac{||\mathrm{e}^{\Lambda t}\boldsymbol{\kappa}_0||_{\mathrm{E}}^2}{||\boldsymbol{\kappa}_0||_{\mathrm{E}}^2} = \max_{||\boldsymbol{\kappa}_0|| \neq 0} \frac{||\mathbb{F}\mathrm{e}^{\Lambda t}\boldsymbol{\kappa}_0||_2^2}{||\mathbb{F}\boldsymbol{\kappa}_0||_2^2} = ||\mathbb{F}\,\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}||_2^2. \tag{18}$$

The quantity $||\mathbb{F}\,\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}||_2$ is the principal singular value σ_1 of the operator $\mathbb{F}\,\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}$ and its computation is straightforward via standard methods,

$$G(t) = \sigma_1^2\big(\mathbb{F}\,\mathrm{e}^{\Lambda t}\mathbb{F}^{-1}\big). \tag{19}$$

This is equivalent to solving the variational problem of maximizing the factor $g(t)$ for a prescribed time t and considering the initial conditions as the degrees of freedom of the problem (Butler and Farrell 1992). The optimal growth $G(t)$ in (19) has been obtained from the linear operator Λ associated with the (n, k) azimuthal–axial mode and for a prescribed positive time t. Therefore, for a fixed set of values Ri, Ro and η, the *maximum energy amplification factor*, $G_{\max}$, is obtained by maximizing $G(t)$ in (19) for all the pairs $(n, k) \in \mathbb{Z} \times \mathbb{R}$ and for $t \in \mathbb{R}^{+}$:

$$G_{\max}(Ri, Ro, \eta) = \sup_{(n,k,t)} G(t). \tag{20}$$

3. Parametric Study of $G_{\max}$

In this section we describe the global features of the growth factor G_{max} defined in (20). The exploration is carried out for the particular case $\eta = 0.881$ and for inner and outer Reynolds numbers in the domain $(Ri, Ro) \in [0, 900] \times [-4000, 500]$, following the specifications of the experimental study provided in Coles (1965). Our attention is mainly focused in the counter-rotating regime, where the flow exhibited subcritical transitions in the laboratory. Nevertheless, for completeness we enhanced our exploration to a small region in the co-rotating regime. We take advantage of the $\mathsf{O}(2)$-symmetry of the problem, i.e., invariance of the system (5)–(10) under axial translations and axial reflections of the form $\{z \to -z, w \to -w\}$, with respect to orthogonal planes to the common axis of the cylinders. The system also has SO(2)-symmetry, i.e., invariance with respect to azimuthal rotations around the centre axis (Chossat and Iooss, 1991). Therefore, we have restricted our computations to the case when both n and k are positive or zero. In this particular study, we have maximized the factor G in (19) for positive times, for azimuthal modes in the range $0 \leq n \leq 15$ and for axial wave numbers in the range $0 \leq k \leq 10$.

In order to validate our numerical results, we have carried out an analysis of the transient growth for axisymmetric disturbances with a fixed axial periodicity. This has been done in order to compare our numerics with the results provided in Hristova *et al.* (2001). In the study carried out by Hristova *et al.*, the distances were nondimensionalized by the length scale $d/2$ and the angular speed ratio was fixed at $\mu = \Omega_o/\Omega_i = -1$. For this particular case, the Reynolds number Re used in Hristova *et al.* (2001) and our inner and outer Reynolds numbers Ri and Ro are related by

$$Ri = 2Re, \qquad Ro = -\frac{2}{\eta} Re. \tag{21}$$

By the same rule, the axial wave number β used in Hristova *et al.* (2001) is related to ours by a factor of two, i.e., $k = 2\beta$. In Figure 1(a), we have represented the transient growth factor for $Ri = 240$, $Ro = -272.42$, $n = 0$ and $k = \pi$, corresponding to the values $Re = 120$ and $\beta = \pi/2$ in Hristova *et al.* The maximum transient growth in this case is $G_{\max} \approx 16.62$, being in very good agreement with Figure 2 of Hristova *et al.* (2001). Nevertheless, the circular Couette flow is linearly unstable in that case for non-axisymmetric perturbations, as seen in Figure 1(b) for $n = 1$. In Figure 1(b), we observe a very similar transient growth which attains a slightly higher maximum value of $G_{\max} \approx 16.66$, although the basic flow eventually exhibits an exponential instability. This justifies a wider study of the transient growth for non-axisymmetric perturbations.

The results of our exploration are summarized in Figure 2. The shaded zone represents the region of the (Ro, Ri)-plane where the circular Couette flow is linearly unstable with respect to axisymmetric or non-axisymmetric perturbations, i.e., $G_{\max} \to \infty$. This region has a lower boundary which is the critical curve where the first linear instability appears. This critical curve has been computed by solving the eigenvalue problem (11) and imposing the condition that the real part of the rightmost eigenvalue of $\mathbb{L}$ be zero. Below the critical boundary prescribed by the modal analysis, the figure shows contours of the function $G_{\max}(Ro, Ri)$. Different features can be pointed out. First, at the bottom right of Figure 2 we have represented the *rigid rotation curve*, $Ri = \eta Ro$, by a dashed line representing the region where both cylinders rotate with the same angular speeds, $\Omega_i = \Omega_o$. We can observe that, close to that region, the Couette flow does not exhibit transient growth. This is clearly visualized in the figure by a narrow stripe containing the rigid rotation curve within which $G_{\max} = 1$. This result is in agreement with previous analyses based on energy methods which concluded that near the rigid rotation region, circular Couette flow is abso-

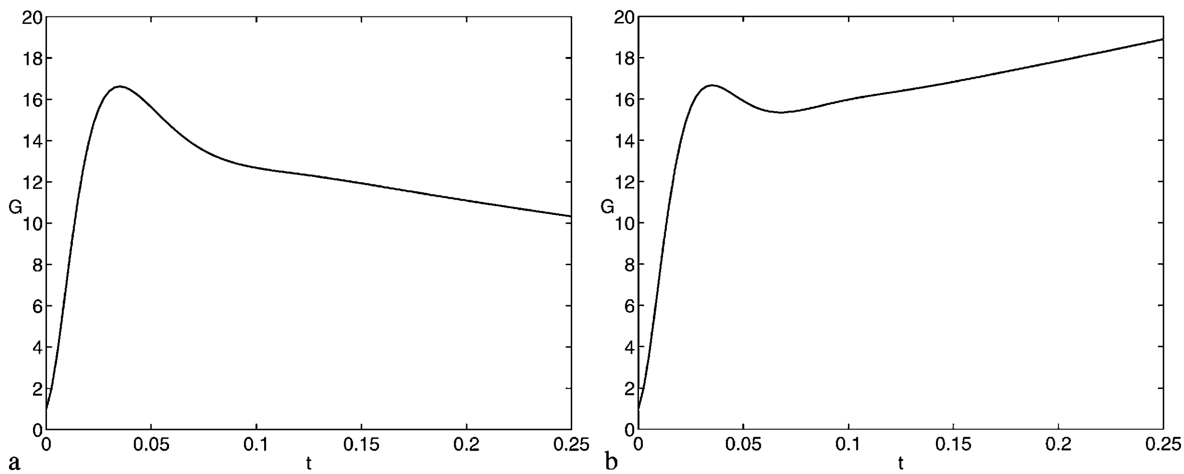

Figure 1. Comparison with simultaneous work (Hristova *et al.*, 2001): (a) Transient growth factor $G(t)$ for $\eta = 0.881$, $Ri = 240$, $Ro = -272.42$, $n = 0$ and $k = \pi$, following Hristova *et al.* for $Re = 120$ and $\beta = \pi/2$. (b) Same computation for $n = 1$.

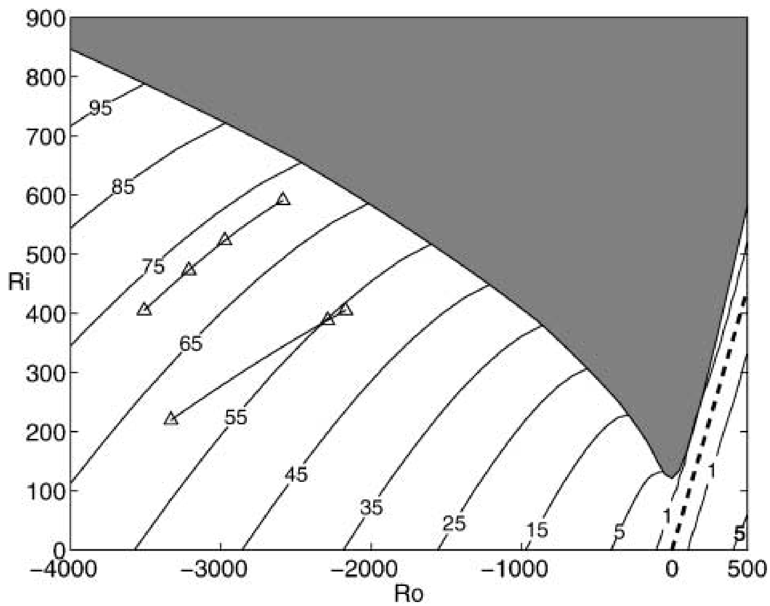

Figure 2. Maximum transient growth factor $G_{\max}$ in the (Ro, Ri)-plane. The dashed line represents the rigid body rotation curve $Ri = \eta Ro$. The lines with white triangles represent the experimental boundaries of transition to turbulence provided in Coles (1965).

Table 1. Parameters for optimal transient growth at the experimental transition points reported by Coles in the upper branch of Figure 2.

Ri	Ro	n	k	$G_{\max}$
591	−2588	10	1.994	71.36
523	−2975	11	1.996	71.58
473	−3213	11	1.920	71.64
405	−3510	11	1.839	71.75

lutely, monotonically and globally stable (Joseph, 1976). Second, in the counter-rotation region, we observe a monotonic growth of $G_{\max}$, which ranges between 1 and 100. This would imply that the energy of any small perturbation would be transiently amplified by almost two orders of magnitude in the counter-rotation region explored in this case. Third, the contours of $G_{\max}$ are *not* tangent to the shaded region over the linear instability boundary. In fact, the intersection is transversal, implying that nonmodal transient growth may still be found slightly above the linear critical values, as reflected in Figure 1(b). Finally, Figure 2 includes the experimental data from Coles (1965). The lines with white triangles represent the experimental boundaries of transition to spiral turbulence reported by Coles above which subcritical transition was found. The two boundaries correspond to two independent experiments carried out with different fluids. In Coles (1965) the discrepancy between the two experimental boundaries was not completely understood. Nevertheless, the upper experimental boundary from Figure 2 is clearly aligned with the contour curves of $G_{\max}$, revealing a correlation between the transition phenomena and the energy amplification factor. We have carried out the computation of $G_{\max}$ at the four points of the upper experimental branch of Figure 2. The optimal values have been included in Table 1. A remarkable fact is that the experimental transition takes place within the range

$$G_{\max} = 71.58 \pm 0.16,$$

with 0.2% relative deviation. This suggests that, although our analysis is only linear, the nonmodal transient growth plays a very important role in the subcritical transition. However, this mechanism is *not* sufficient for the eventual development of spiral turbulence.

4. Conclusions

Transient linear effects in various flows have been studied in recent years, but there has not been much attention of this kind to Taylor–Couette flows. Here we have provided comprehensive exploration of the optimal transient growth in the counter-rotating Taylor–Couette problem. Significant energy transient growth has been found in the linearly stable regime of counter-rotation. The numerical computations of the maximum amplification factor are consistent with the experimental threshold values obtained by Coles. Non-axisymmetric modes seem to be more effective in the transient mechanism and axisymmetric azimuthal streaks may still be observed as well, although they exhibit a weaker amplification. Direct numerical simulation of the problem would be required in order to understand how these linear effects combine with nonlinear ones to bring about transition to turbulence.

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